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**STUDY OF FLUTTER RELATED COMPUTATIONAL PROCEDURES  
FOR MINIMUM WEIGHT STRUCTURAL SIZING OF ADVANCED  
AIRCRAFT - SUPPLEMENTAL DATA**

by R.F. O'Connell, H.J. Hassig, N.A. Radovich

Prepared by

**LOCKHEED-CALIFORNIA COMPANY**

BURBANK, CALIFORNIA

for Langley Research Center

**NATIONAL AERONAUTICS AND SPACE ADMINISTRATION**



**AUGUST 1975**

STUDY OF FLUTTER RELATED COMPUTATIONAL PROCEDURES FOR  
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By R. F. O'Connell, H. J. Hassig and N. A. Radovcich,  
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## ABSTRACT

The subject study addressed computational aspects of (1) flutter optimization (minimization of structural mass subject to specified flutter requirements), (2) methods for solving the flutter equation, and (3) efficient methods for computing generalized aerodynamic force coefficients in the repetitive analysis environment of computer-aided structural design. The principal results of the study are summarized in a companion report. The present report contains supplemental data and supporting information on various aspects of the study including the following:

Details of a two-dimensional Regula Falsi approach to solving the generalized flutter equation are presented.

The method of Incremented Flutter Analysis and some of its applications are discussed.

The use of Velocity Potential Influence Coefficients in a five-matrix product formulation of the generalized aerodynamic force coefficients is delineated. Options for computational operations required to generate generalized aerodynamic force coefficients are compared in detail.

Theoretical considerations related to optimization with one or more flutter constraints are presented as well as practical experience with an actual structural design problem.

Expressions for derivatives of flutter-related quantities with respect to design variables are presented. Included are flutter-speed derivatives with variable modes.

A bibliography is included.



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## SYMBOLS AND DEFINITIONS

$[ \quad ]$	square, rectangular matrix
$[ \quad ]^T$	transpose of a matrix
$\{ \quad \}$	column matrix
$[ \quad ]$	row matrix
$\begin{bmatrix} & \\ & \end{bmatrix}$	diagonal matrix
$  \quad  $	determinant
$[X_1]$	matrix $[X]$ with last column omitted
$[\underline{X}]$	matrix $[X]$ with last row omitted
$A$	constant multiplying the additive column of design variable increments (equation 5.17)
$a_n, a_{n+1}$	amplitudes of successive cycles
$[A(ik)]$	aerodynamics matrix (function of $k$ and Mach number), modalized aerodynamics matrix
$[A(p)]$	aerodynamics matrix (function of $p$ and Mach number)
$[\tilde{A}(p)]$	$\frac{1}{2}\rho V^2[A(p)]$
$[\bar{A}_j]$	$j = 0,1,2,3$ ; defines $[A(k)]$ in cubic spline interpolation (equation 4.37)
$A_{ij}, A_{ij}(k)$	generalized aerodynamic force coefficients
$[AIC], [AIC(k)]$	basic aerodynamics influence coefficients (function of $k$ and Mach number) defined by equation 4.1
$[\overline{AIC}_j]$	$j = 0,1,2,3$ ; defines $[AIC(k)]$ in cubic spline interpolation (similar to equation 4.37)
$[\overline{AW}_j]$	$j = 0,1,2,3$ ; defines $[AW(k)] = [AIC(k)][W(k)]$ in cubic spline interpolation (similar to equation 4.37)
$B$	constant multiplying the subtractive column of design variable increments (equation 5.17)

$[B(k)]$	$[K]^{-1} \left[ \frac{1}{c^2} [M] k^2 + \frac{1}{2} \rho [A(ik)] \right]$
b	wing span
C	constant
$C_i$	ratio between $m_i$ and $P_i$ : $m_i = C_i P_i$ (Section 5.2.3)
c	reference chord
$D( )$	flutter determinant
$[D]$	viscous damping matrix
$[D_j]$	matrix relating control system displacements to structural displacements
$[DPHX]$	differentiating matrix
$[DX]$	interpolation and differentiation matrix relating slopes at downwash collocation points to displacements at structural nodes (equation 4.13)
$[DZ]$	interpolation matrix relating translations at downwash collocation points to displacements at structural nodes (equation 4.13)
$[EH]$	extrapolating matrix
f	frequency, Hz
g	structural damping
$[H]$	interpolation matrix relating displacements at lumped aerodynamic load points to displacements at structural nodes (equation 4.17)
$[\overline{HA}]_j$	$j = 0,1,2,3$ ; defines $[HA(k)] = [H]^T [AIC(k)]$ in cubic spline interpolation (similar to equation 4.37)
$[\overline{HAW}]_j$	$j = 0,1,2,3$ ; defines $[HAW(k)] = [H]^T [AIC(k)] [W(k)]$ in cubic spline interpolation (similar to equation 4.37)
$H_j(p)$	transfer function of automatic control system (function of p)

$h_i$	displacement at aerodynamic load point in deflection mode $i$
$ID(x,y)$	imaginary part of $D(x,y)$
$[IPHX]$	interpolating matrix
$[K]$	stiffness matrix, modalized stiffness matrix
$[K_0]$	base stiffness matrix
$[\Delta K_i]$	incremental stiffness matrix per unit design variable $i$
$[\tilde{K}]$	$(1 + ig)[K]$ , stiffness matrix
$k$	reduced frequency $k = \frac{\omega c}{V}$
KEAS	knots equivalent airspeed
$\mathcal{L}_\ell(k)$	polynomial multipliers used in Lagrange's interpolation formula
$[M]$	mass matrix, modalized mass matrix
$[\tilde{M}]$	$\frac{V^2}{c^2} [M]$ , mass matrix
$[M_0]$	base mass matrix
$[\Delta M_i]$	incremental mass matrix per unit design variable $i$ .
$M$	total mass associated with the design variables
$m_i$	design variable associated with structural mass (in mass or weight units)
$P_i$	general design variable
$p$	complex root of flutter equation for a given flight condition ( $p = (\gamma + i)k$ )
$p_j$	aerodynamic lifting pressure distribution corresponding to deflection mode $j$

$q$	modal degrees of freedom (modal participation coefficients)
$\{q\}$	modal column corresponding to solution of characteristic equation
$RD(x,y)$	real part of $D(x,y)$
$[r]$	modal row corresponding to solution of characteristic equation
$T(x,y)$	defined by equation 2.20
$V$	speed, flutter speed
$V_R$	required flutter speed (minimum allowable flutter speed: 1.20 $V_D$ for commercial, 1.15 $V_L$ for military)
$V_U$	unsatisfactory flutter speed
$V_D$	design speed according to Federal Aviation Regulations
$[VPIC]$	matrix of velocity potential influence coefficients
$W_1$	total mass addition associated with the additive column of design variable increments (equation 5.21)
$[W_1], [W_2], [W_3], [W_4]$	row matrices for numerical integration
$[W], [W(k)] = [DX] + ik[DZ]$	angle-of-attack generating matrix (function of $k$ ) (equation 4.16)
$[WF], [WFD]$	generalized weighting matrices (Section 4.1.2)
$x$	coordinate in a fore-and-aft direction, general variable
$y$	coordinate in a lateral direction, general variable
$\{Z\}$	column matrix of aerodynamic forces
$\{z\}$	column matrix of displacements of structural nodes
$[Z]$	matrix of modal columns of displacements of structural nodes
$\alpha$	angle of attack
$\beta_i$	general design variable

$\gamma$	normalized real part of $p = (\gamma + i) k$
$\lambda$	$(1 + i g) V^2$ , Lagrangian multiplier
$\lambda_1, \lambda_2$	Lagrangian multipliers
$\rho$	air density
$\phi$	aerodynamic velocity potential
$\omega$	circular frequency, rad/s
'	indicates derivative of a one-variable function
T.E.	trailing edge
in-flight mode:	modal column corresponding to a characteristic solution of the flutter equation
flutter mode:	in-flight mode that becomes unstable within the velocity range considered
hump mode:	in-flight mode with a minimum damping point within the velocity range considered
Identification of count formulas XYyZ (Section 4.3.2)	
X = Oper	total number of operations
X = Stor	number of matrix element words to be stored for easy access
X = Cor	number of matrix element words to be kept in core for interpolation
X = Read	number of matrix element words to be read into core for interpolation when switching from one k interval to the next
Y	sequence of multiplication options (equations 4.40 - 4.43)
y = a	$[W(k_l)]$ is input
y = b	$[DX]$ and $[DZ]$ are input
Z = Poly	cubic polynomial interpolation
Z = Spline	cubic spline interpolation



The interpretation of the matrix dimensions used is as follows:

M: number of modes used in flutter analysis

N: number of discrete degrees of freedom: structural displacements

K: number of lumped aerodynamic forces

D: number of downwash collocation points

$\phi$ : fraction of non-zero elements in rows of  $[W(k)]$  and  $[H]$

In addition:

$n_\delta$ : number of  $k$  intervals needed

$n_\lambda$ :  $n_\delta + 3$

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1. INTRODUCTION

In a companion report (Reference 1) the authors have presented the results of the present study that are in most direct response to the objectives of the contract. To generate the results in Reference 1, however, considerable supporting work was performed that has potential significance for the worker in the field of structural optimization. The present report summarizes a major portion of that supporting work.

The two-dimensional Regula Falsi approach to solving the generalized flutter equation is presented in considerable detail including some experience with a frequency and velocity sweep method for a global search for flutter frequency and speed.

The method of Incremented Flutter Analysis and some of its applications is presented.

In a section on aerodynamics a method is presented for the use of velocity potential aerodynamic influence coefficients in the basic five matrix product formulation of the generalized aerodynamic force coefficients. A method to prevent oscillatory non-convergence, "hunting", in the flutter solution due to cubic polynomial interpolation of the aerodynamics is presented. Options for computational operations required to generate generalized aerodynamic force coefficients are compared in detail.

Section 5 is an attempt to approach flutter optimization with the help of elementary considerations. The authors believe this section can aid the worker in the field of flutter optimization in improving methods of analysis.

Practical experience with an ad hoc method of optimization is presented. It indicates that the number of modal degrees of freedom required for a moderately complicated arrow wing supersonic transport design may be considerably higher than the number used in the numerical examples found in the literature.

Expressions for the derivatives of the in-flight modal damping with respect to the design variables were derived during the study. They are presented, together with other derivatives, in Section 7.

The report concludes with a bibliography.

## 2. TWO-DIMENSIONAL REGULA FALSI

One of the objectives of the present study is to survey and evaluate methods of solving the flutter equation which are suitable for repetitive use in a structural resizing procedure with flutter constraints. Several methods have been evaluated and the results are presented in Reference 1. The two-dimensional Regula Falsi approach was chosen for further development since it was considered most promising after an overall engineering evaluation, including the generation of some numerical data.

In the following sections the basic mathematical formulation of the method is developed and the generalized flutter equation to which it is applied is discussed. One section is devoted to frequency and velocity sweep, a global search method for flutter speed and frequency that was partly developed as a result of two-dimensional Regula Falsi investigations. Finally the Lockheed-California Company's two-dimensional Regula Falsi program as developed for use in a flutter optimization is discussed.

### 2.1 Basic Mathematical Formulation

The Regula Falsi and Newton's method are related iterative methods of solving the non-linear equation  $f(x) = 0$ . The basic Regula Falsi is shown in Figure 2-1. The function  $f(x)$  is evaluated at two trial values of  $x$ ,  $x_1$  and  $x_2$ , leading to  $f_1$  and  $f_2$ . By linear interpolation or extrapolation a new value of  $x$ ,  $x_3$ , is found with the associated  $f_3$ . Next  $f_3$  and  $f_2$  are used to generate  $x_4$ . In a variation of the method, here called Type II, the interpolation or extrapolation is done with the latest value of  $f$  and the smallest of all preceding  $f$ 's (Figure 2-2)

The basic Newton Method is shown in Figure 2-3. The function  $f(x)$  and its derivative  $f'(x)$  are evaluated at a trial value  $x_1$ , leading to  $f_1$  and  $f'_1$ . The tangent to the curve at  $x_1$  is intersected with the abscissa leading to a new value of  $x$ ,  $x_2$ , after which the procedure is repeated. The derivative  $f'(x)$  can be determined analytically or by means of a finite difference technique. In the latter form Newton's method resembles the Regula Falsi method even more than in its basic form.

Figure 2-4 illustrates how a poor initial trial  $x_1$  may lead the iteration process away from the desired solution in the case of the basic Newton method. The finite difference form of Newton's method might have done better, especially if a relatively large value of  $\Delta x$  were initially chosen.

Obviously the Regula Falsi also can have convergence troubles. However, by trying to choose the values of  $x_1$  and  $x_2$  such that they tend to straddle the solution these convergence troubles are minimized.

Both the Newton method and the Regula Falsi can be expanded to  $n$  equations with  $n$  unknowns, but, even for  $n=2$ , a graphical representation

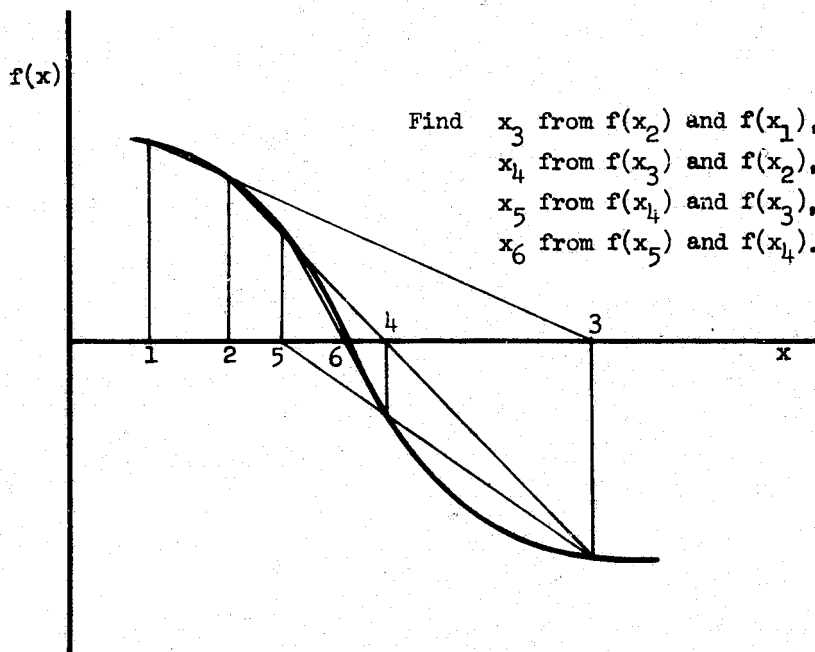


Figure 2-1: Regula Falsi - Type I.

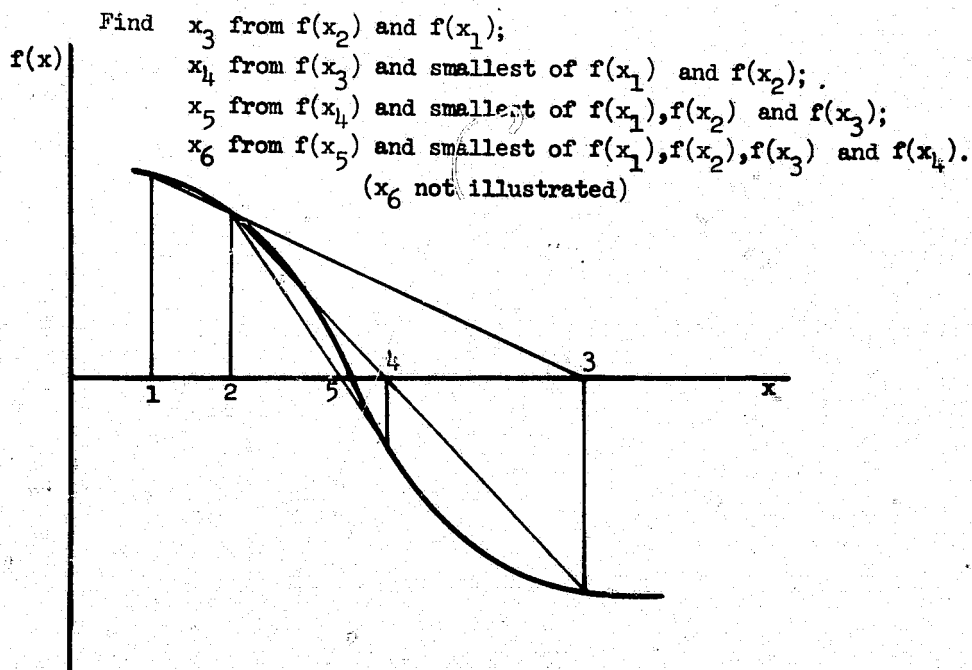


Figure 2-2: Regula Falsi - Type II.

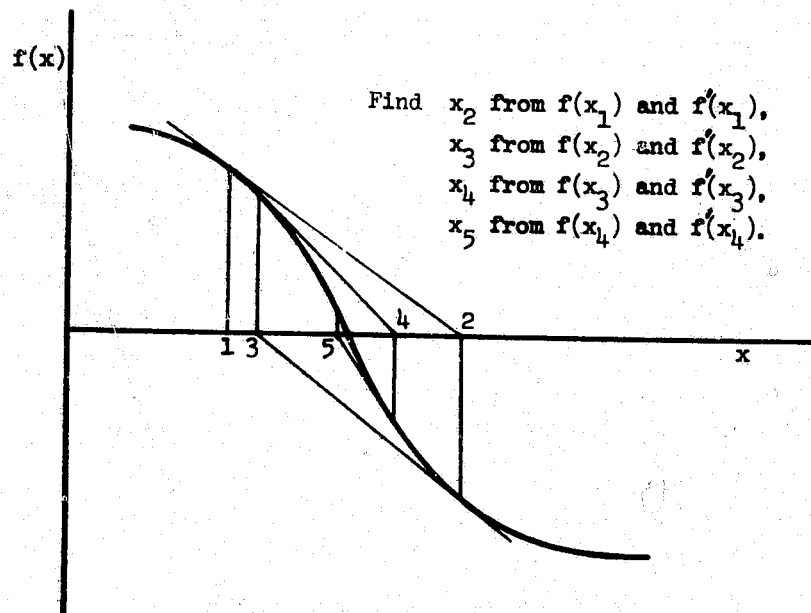


Figure 2-3: Newton's Method.

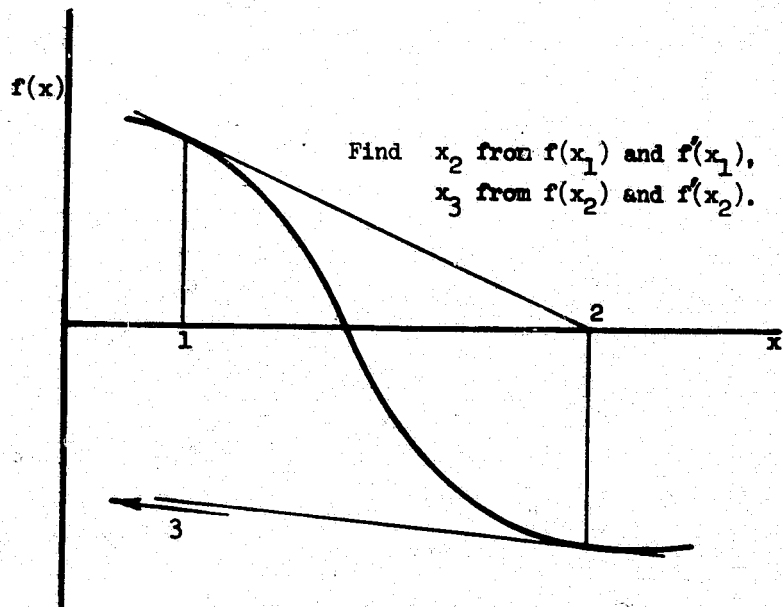


Figure 2-4: Newton's Method - Unsuccessful.

of the procedures is almost impossible. During the present contract, it was found that for  $n=2$  the type of non-convergence suggested by Figure 2-4 occurred rather frequently, even when the Regula Falsi was used.

In the following the Regula Falsi is presented in an extension to two equations with two unknowns. In this form it is referred to as the two-dimensional Regula Falsi.

Let

$$F(x,y) = 0 \quad \text{and} \quad G(x,y) = 0 \quad (2.1)$$

be the two equations with the two unknowns  $x$  and  $y$ . Let  $x_i, y_i$  ( $i=1,2,3$ )

be three pairs of trial values in the vicinity of the solution to the equations 2.1. Define a plane  $f(x,y) = a_f x + b_f y + c_f$  containing the points

$F(x_i, y_i)$  ( $i=1,2,3$ ) and a plane  $g(x,y) = a_g x + b_g y + c_g$  containing the points  $G(x_i, y_i)$ .

The coefficients  $a_f, b_f$  and  $c_f$  are determined by the matrix equation:

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \begin{Bmatrix} a_f \\ b_f \\ c_f \end{Bmatrix} = \begin{Bmatrix} F(x_1, y_1) \\ F(x_2, y_2) \\ F(x_3, y_3) \end{Bmatrix} \quad (2.2)$$

In an obvious shortened notation:

$$\begin{Bmatrix} a_f \\ b_f \\ c_f \end{Bmatrix} = [XY]^{-1} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \quad (2.3)$$

Similarly,

$$\begin{Bmatrix} a_g \\ b_g \\ c_g \end{Bmatrix} = [XY]^{-1} \begin{Bmatrix} G_1 \\ G_2 \\ G_3 \end{Bmatrix} \quad (2.4)$$

Figure 2-5 illustrates the procedure. The solution  $x_4, y_4$  to the equations  $f(x, y) = 0$  and  $g(x, y) = 0$  is taken as the first approximation to the equations 2.1. This leads to the equations

$$a_f x_4 + b_f y_4 = -c_f \quad (2.5)$$

$$a_g x_4 + b_g y_4 = -c_g \quad (2.6)$$

from which:

$$\begin{Bmatrix} x_4 \\ y_4 \end{Bmatrix} = - \begin{bmatrix} a_f & b_f \\ a_g & b_g \end{bmatrix}^{-1} \begin{Bmatrix} c_f \\ c_g \end{Bmatrix} \quad (2.7)$$

The process is repeated using the pairs  $x_4, y_4$  and two of the pairs  $x_i, y_i$  ( $i=1,2,3$ ), eliminating the pair that least satisfies equation 2.1 (e.g. the pair with the largest values of  $|F(x_i, y_i)| + |G(x_i, y_i)|$ ). It is continued until convergence is reached.

Characteristic of the two-dimensional Regula Falsi is:

1. three pairs of trial values are required to initiate the process
2. the two functions, but no derivatives, must be evaluated once at each step, except at the first step for which the two functions must be evaluated for three pairs of values  $x, y$

## 2.2 The Generalized Flutter Equation

When using the  $k$  method the flutter equation can be written as:

$$\left[ -\frac{1}{c^2} [M] k^2 + \frac{1+ig}{v^2} [K] - \frac{1}{2} \rho [A(ik)] \right] \{q\} = 0 \quad (2.8)$$

One of several possible methods of solving this equation is to determine the characteristic value  $\lambda = \frac{1+ig}{v^2}$  for several values of the reduced frequency  $k = \frac{\omega c}{v}$ , keeping all other quantities in the equation constant (Reference 2).



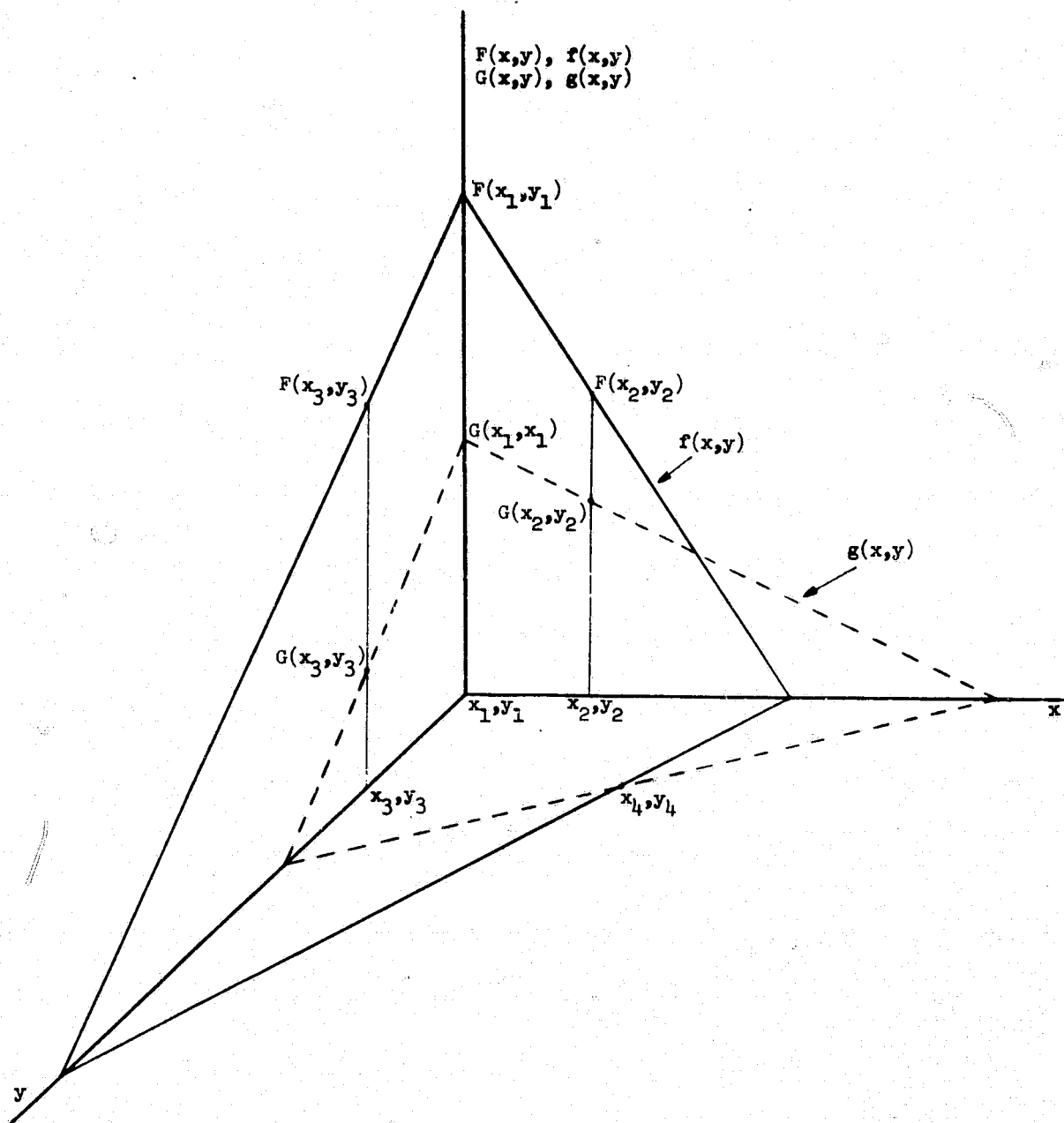


Figure 2-5: Illustration of Two-Dimensional Regula Falsi.

In the p-k method the flutter equation is

$$\left[ \frac{V^2}{c^2} [M] p^2 + (1+ig) [K] - \frac{1}{2} \rho V^2 [A(ik)] \right] \{q\} = 0 \quad (2.9)$$

and solutions  $p=(\gamma+i)k$  are sought for selected combinations of values of  $V$  and  $\rho$ , such that  $[A(ik)]$  is evaluated for the value of  $k$  that defines the imaginary part of the solution. The p-k method formulation is convenient for the inclusion of viscous damping and control system transfer functions (Reference 3). This is accomplished by writing:

$$\left[ \frac{V^2}{c^2} [M] p^2 + \frac{V}{c} [D] p + (1+ig) [K] - \frac{1}{2} \rho V^2 [A(ik) - \sum H_j(p) [D_j]] \right] \{q\} = 0 \quad (2.10)$$

where  $H_j(p)$ ,  $j=1,2,\dots$ , represents transfer functions of the control system and  $[D_j]$  relates control system displacements to the structural displacements;  $[D]$  is a viscous damping matrix.

A further generalization of the flutter equation can be made by making the stiffness matrix and the inertia matrix functions of design variables  $m_i$ , which is the standard procedure for structural optimization. In addition, other quantities, such as  $[D]$ ,  $[D_j]$ , as well as transfer function coefficients in  $H_j(p)$  may be made functions of design variables.

Equation 2.10 implies that the determinant of the square matrix on the left hand side is zero and thus, in a very general form, the flutter equation can be written as:

$$D \{(\gamma+i)k, g, V, \rho, m_i\} = 0 \quad (2.11)$$

The quantity  $D$  is called the flutter determinant. For arbitrary values of the variables it has a complex value. Thus equation 2.11 represents two equations equivalent to equations 2.1. In principle they can be solved for two unknowns for given values of the other variables.

Letting  $\gamma = 0$  and solving for  $g$  and  $V$  corresponds to the traditional k method of solving the flutter equation. Equation 2.8 can be written in the form

$$[\lambda [1] - [B(k)]] \{q\} = 0 \quad (2.12)$$

and can be solved for  $\lambda = \frac{1+ig}{V^2}$  by the classical power method and improvements thereof.

Solving for  $(\gamma+i)k$  corresponds to the p-k method. According to Reference 3, equation 2.10 for (small) values of  $\gamma$  of interest is "almost" an analytic function of  $p = (\gamma+i)k$  and it can be solved by determinant iteration which is basically a one-dimensional Regula Falsi method.

In the k method and the p-k method the flutter speed is determined indirectly by interpolation between solutions for several values of k or several values of V. However, equation 2.11 can be solved directly for the flutter speed at a given value of the structural damping g by letting  $\gamma = 0$  and solving for the two unknowns k and V.

Solving equation 2.11 for k and one of the design variables, assuming all other variables fixed, has found application in structural optimization with flutter constraints (Reference 1). It forms the basis for the method of Incremented Flutter Analysis which is discussed in Section 3.

Since equation 2.10 actually is not an analytic function of  $p = (\gamma+i)k$  it should be considered as two equations with the two independent unknowns  $\gamma$  and k. Lockheed's program for solving the flutter equation according to the p-k method has indeed an option to use the two-dimensional Regula Falsi. During numerical testing it was found to be slightly less efficient than determinant iteration in that on the average one more iteration step is required.

The two-dimensional Regula Falsi Program used in conjunction with the p-k method is a general purpose multi-dimensional Regula Falsi program. Initially it was used unchanged to solve the generalized flutter equation for k and V, and for k and  $m_1$ . Few convergence difficulties were encountered during an optimization study of an arrow wing supersonic transport when solving for k and  $m_1$ . Initially, however, convergence difficulties occurred erratically when solving directly for flutter speed and flutter frequency. To gain an understanding of the problem the determinantal flutter equation for several fixed configurations was written as

$$D(\omega, V) = 0 \quad (2.13)$$

and its behavior as a function of  $\omega$  and V was studied in depth. This led to an improved version of the two-dimensional Regula Falsi program, described in Section 2.4, and had as a by-product a potential global search method for estimating flutter speed and flutter frequency which is discussed in the next section.

## 2.3 Frequency and Velocity Sweep

As indicated in the previous section initial numerical experiments with the two-dimensional Regula Falsi ran into convergence difficulties when solving directly for flutter speed and frequency. To gain an understanding of the problem the flutter determinant,  $D(\omega, V)$ , for a fixed configuration was evaluated for many values of  $\omega$  and  $V$ . Various types of plots relating  $\omega$ ,  $V$  and the real and imaginary part of the determinant were studied. One type of plot, the argument (or phase) of the flutter determinant versus frequency and versus velocity, led to results that may be of value in developing a global search for approximate values for the flutter speed and flutter frequency. These approximate values could subsequently be used in a local search by the two-dimensional Regula Falsi method to determine accurate solutions. Since in most approaches to flutter optimization it is assumed that approximate solutions of the flutter equation are available to initiate a new solution process this global search method was not pursued in the present study. The results obtained thus far are summarized in this section.

**2.3.1 Frequency Sweep.** It was found that if the complex flutter determinant is evaluated for a speed  $V_1$  which is less than the lowest flutter speed the values of  $\omega$  that satisfy the equation  $\arg \{D(\omega, V_1)\} = \frac{\pi}{2}$  are good approximations for the flutter frequencies. Three sample cases are presented and discussed.

All three cases are for symmetric flutter analyses of the arrow wing transport shown in Figure 6-1 at a Mach number  $M = 0.9$  and an operating weight empty (OWE) of 321,000 lbs. Further details regarding structural representation, structural and vibrational degrees of freedom, and aerodynamics are presented in Section 6.1. For all cases the structural damping  $g = 0.02$ . Natural vibration modes, including two zero frequency modes, are used in the flutter analysis.

Figure 2-6 is the  $f$ - $g$ - $V$  diagram for Case 1. It results from an analysis with 20 modes and variable air density (matched atmospheric conditions). The flutter frequencies and associated speeds are 2.07 Hz, 318 KEAS and 2.44 Hz, 375 KEAS. Figure 2-7 shows the argument of the flutter determinant as a function of the frequency for several values of  $V$ . It can be seen that only those flutter frequencies are detected that correspond to a flutter speed above the value of  $V$  for which the argument is determined.

Figure 2-8 is the  $f$ - $g$ - $V$  diagram for Case 2. It results from an analysis with 20 modes and a constant air density corresponding to 18,000 ft. To confirm the true character of the fourth and the fifth mode, solutions for these modes between 400 and 420 KEAS were determined at very small velocity intervals (Figure 2-9). The flutter frequencies and associated flutter speeds are 1.15 Hz, 542 KEAS; 2.71 Hz, 390 KEAS and 4.21 Hz, 609 KEAS. Figure 2-10 shows the frequency sweep: argument of flutter determinant versus frequency

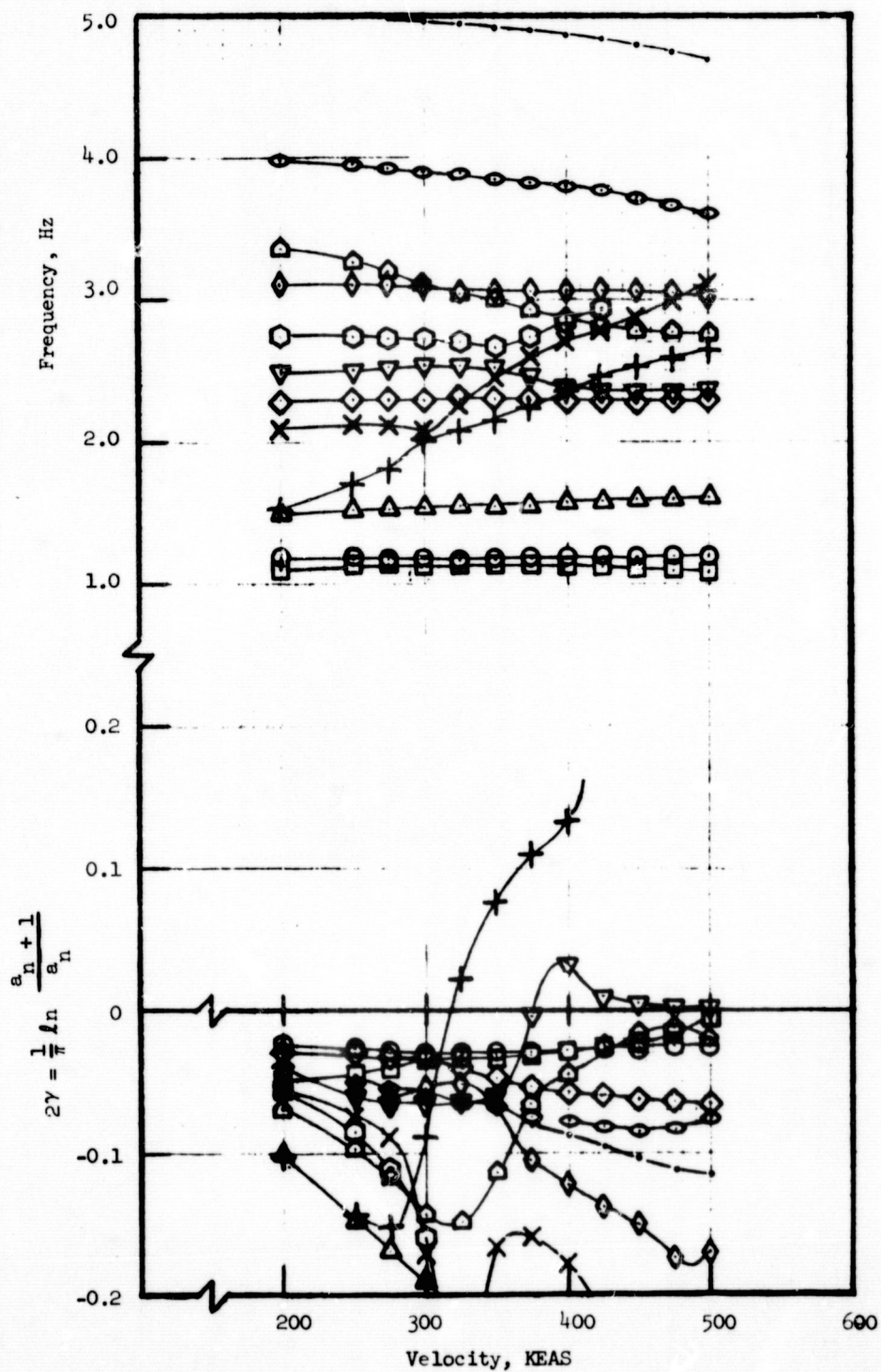


Figure 2-6: Case 1; f-g-V Diagram.

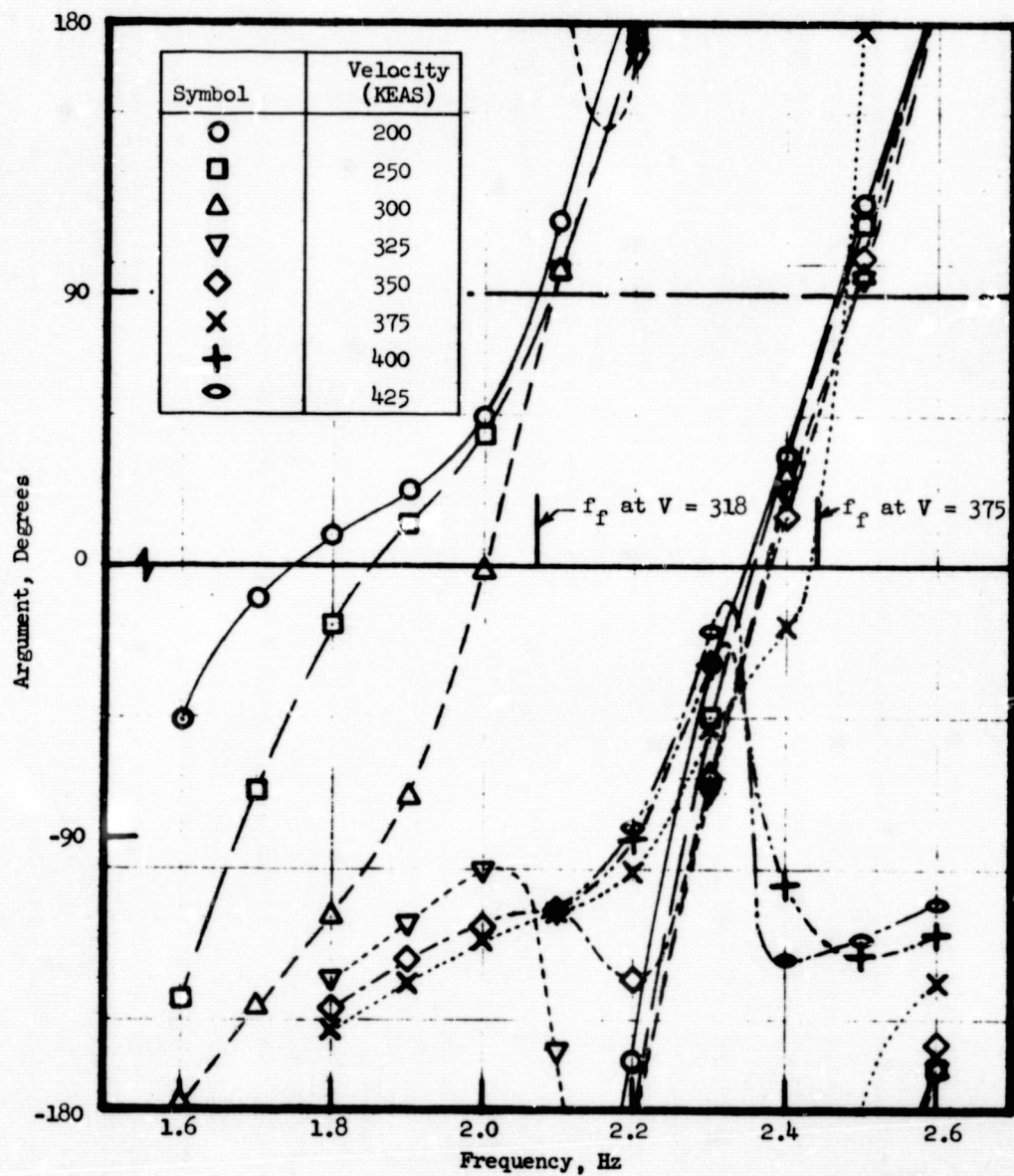


Figure 2-7: Case 1; Frequency Sweep for Several Speeds;  $f_f$  is Flutter Frequency Obtained from Figure 2-4.



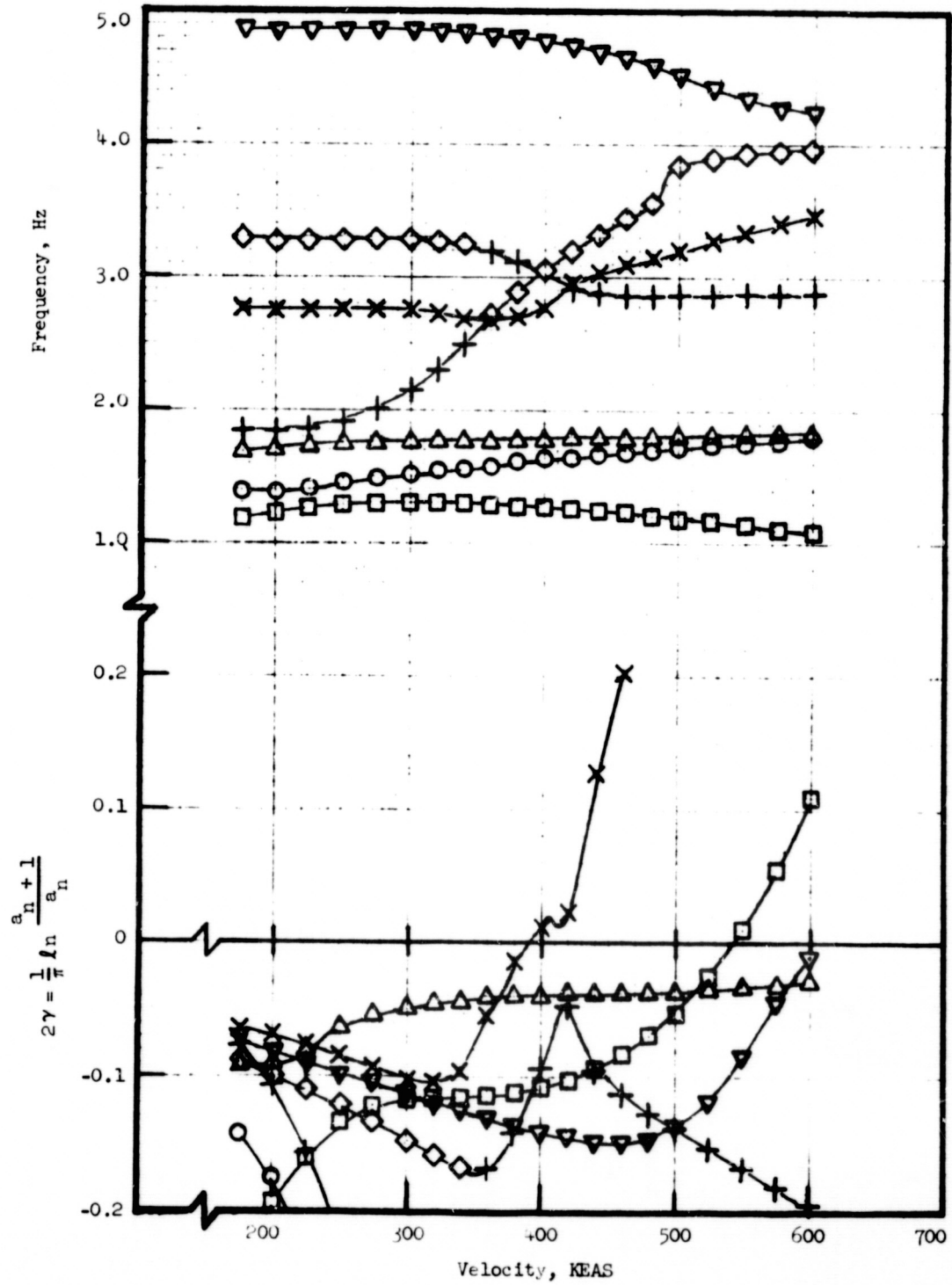


Figure 2-8: Case 2; f-g-V Diagram.



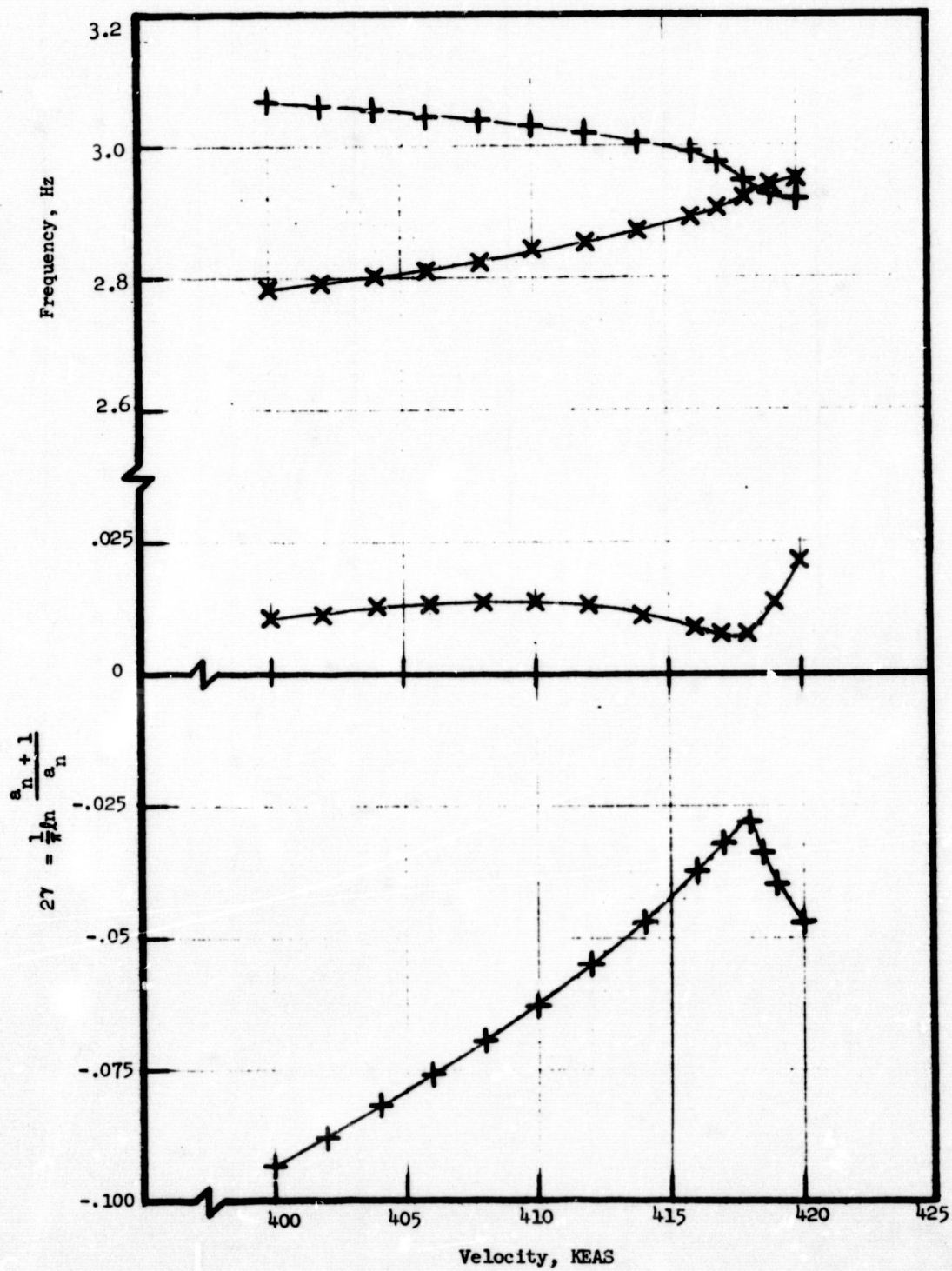


Figure 2-9: Case 2; Detail of f-g-V Diagram.

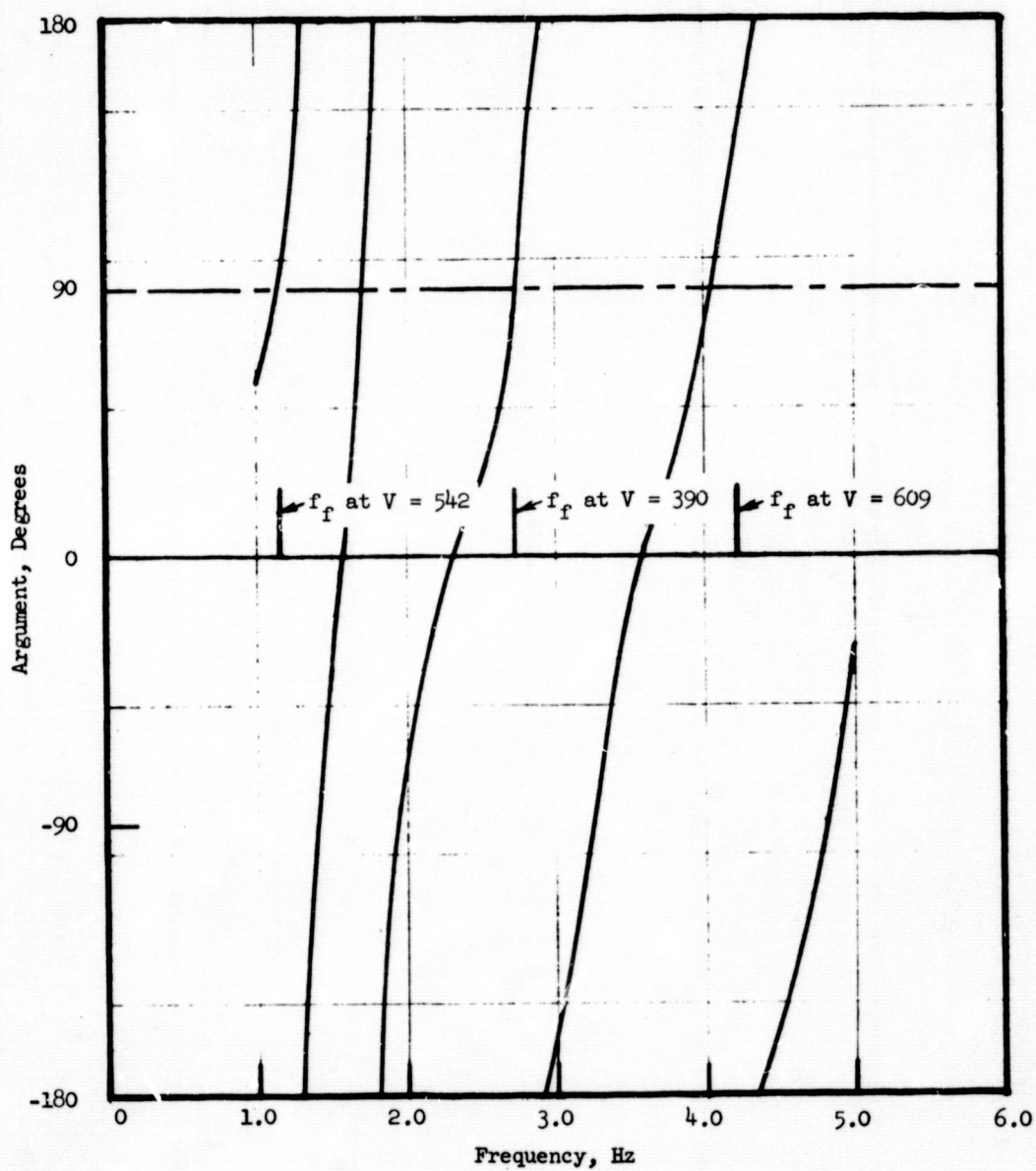


Figure 2-10: Case 2; Frequency Sweep for 250 KEAS;  $f_f$  is Flutter Frequency Obtained from Figure 2-8.

for  $V = 250$  KEAS. The three flutter frequencies are detected. Another flutter frequency  $f \approx 1.70$  Hz is indicated. It is believed if the  $f$ - $g$ - $V$  diagram were extended to higher velocities a flutter speed corresponding to 1.70 Hz would be found.

The  $f$ - $g$ - $V$  diagram for a third case is shown in Figure 2-11. This case is different from Case 2 in that only 12 modes are used. This  $f$ - $g$ - $V$  diagram is very similar to the one in Figure 2-8. The key difference is that the curve corresponding to the lowest flutter speed in Figure 2-8 has an S-curve above the zero damping line and continues steeply into the unstable region, whereas in Figure 2-11 the corresponding curve turns stable and becomes a hump mode.

The zero damping points in Figure 2-11 occur at 2.71 Hz and 410 KEAS, 3.05 Hz and 450 KEAS, 1.14 Hz and 552 KEAS and, on the back side of the hump mode, 2.85 Hz and 451 KEAS.

The frequency sweep corresponding to Figure 2-11 is shown in Figure 2-12; it is very similar to Figure 2-10. The flutter frequencies corresponding to the backside of the hump mode and the steeply unstable mode, 3.05 Hz and 2.85 Hz respectively, are not detected. Additional cases very similar to Case 3 were investigated with similar results.

A preliminary conclusion is that the frequency sweep as defined above can be used for a global search for flutter frequencies. However, there is no assurance that all flutter frequencies are detected. Further investigation may lead to a useful method.

**2.3.2 Velocity Sweep.** It was found that if the complex flutter determinant is evaluated for a frequency  $\omega_1$  which is a little less than a flutter frequency the value  $V$  that satisfies the equation  $\arg \{D(\omega_1, V)\} = -\frac{\pi}{2}$  is a good approximation for the associated flutter speed.

Figure 2-13 shows the argument of the flutter determinant for Case 1 as a function of the velocity for several values of  $f$ . The highest value of  $f$  that is still below the estimated flutter frequency according to Figure 2-7 is  $f = 2.00$  Hz. For the corresponding value of  $\omega$ ,  $\arg \{D(\omega_1, V)\} = -\frac{\pi}{2}$  is satisfied for  $V = 319$  KEAS. The actual flutter speed is 318 KEAS (Figure 2-6).

A velocity sweep for Case 2 is shown in Figure 2-14 for  $f = 2.65$  Hz, a little below the estimated flutter frequency of 2.73 Hz indicated in Figure 2-10. No intersection with the  $-90^\circ$  argument line occurs. A velocity sweep for Case 3, for  $f = 2.70$  Hz, is shown in Figure 2-15;  $f = 2.70$  Hz is a little below the estimated flutter frequency of 2.77 Hz indicated in Figure 2-12. The intersection with the  $-90^\circ$  argument line indicates a flutter speed of 412 KEAS which compares with 410 KEAS as obtained from the  $f$ - $g$ - $V$  diagram (Figure 2-11).



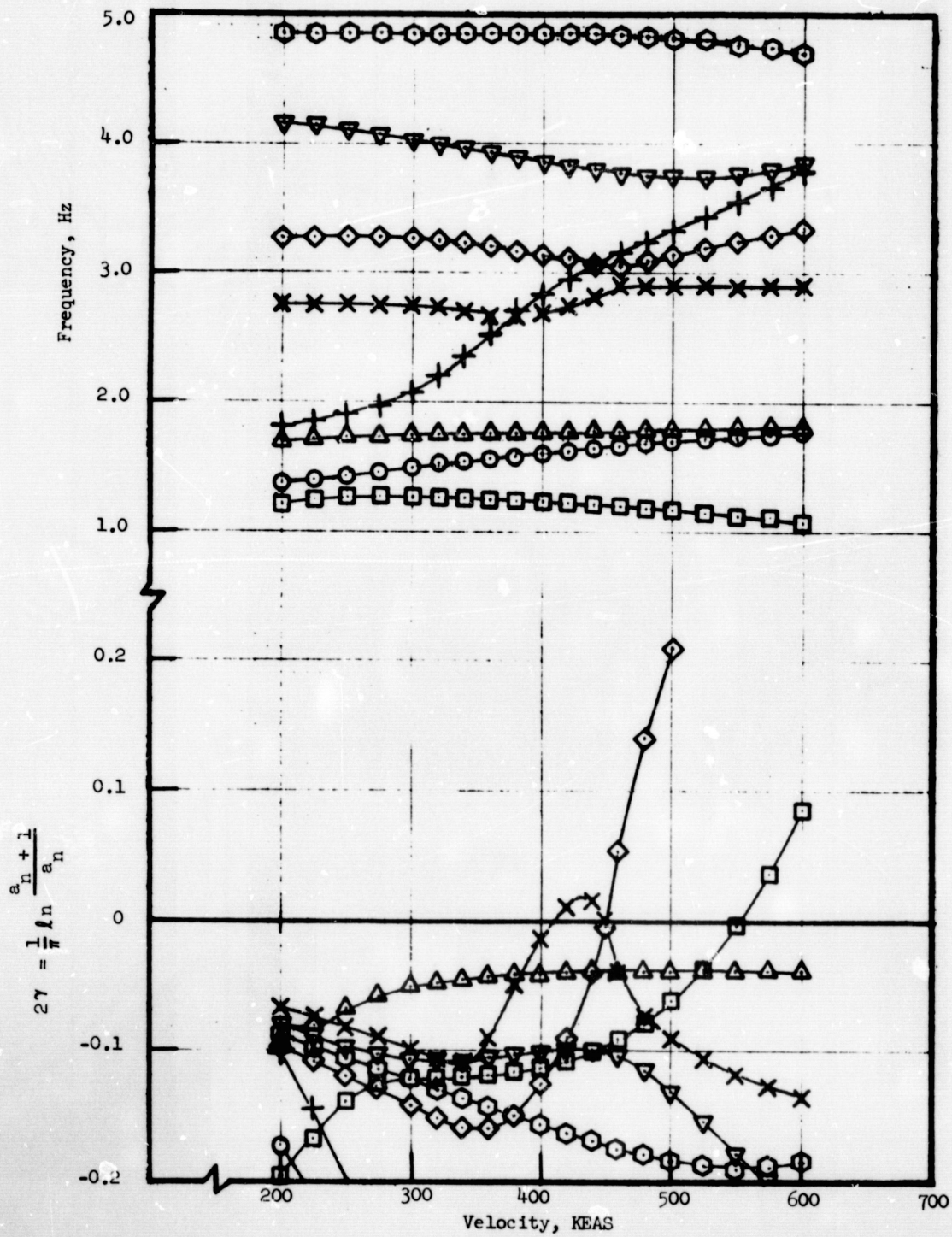


Figure 2-11: Case 3; f-g-V Diagram.

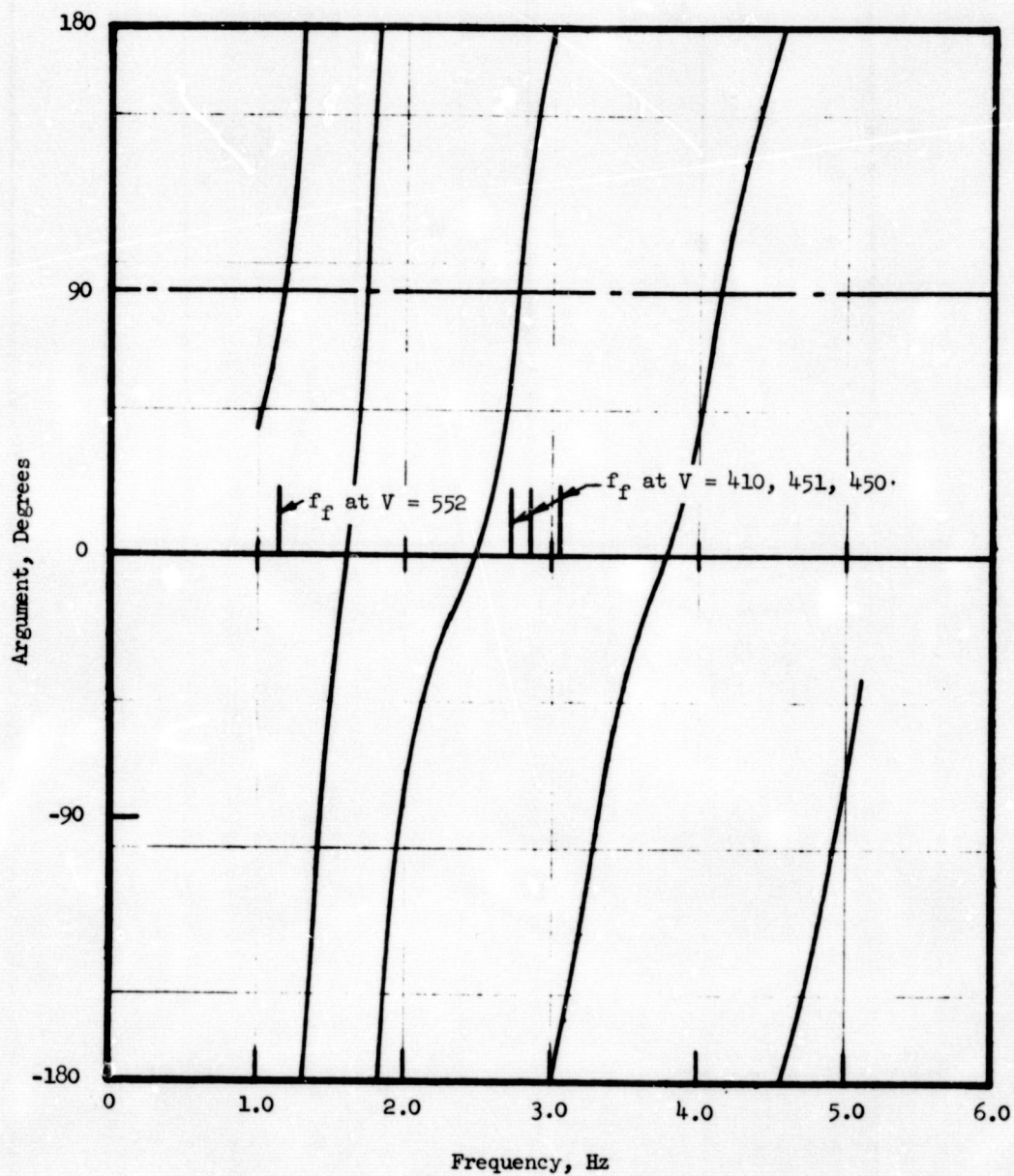


Figure 2-12: Case 3; Frequency Sweep for 250 KEAS;  $f_f$  is Flutter Frequency Obtained from Figure 2-11.

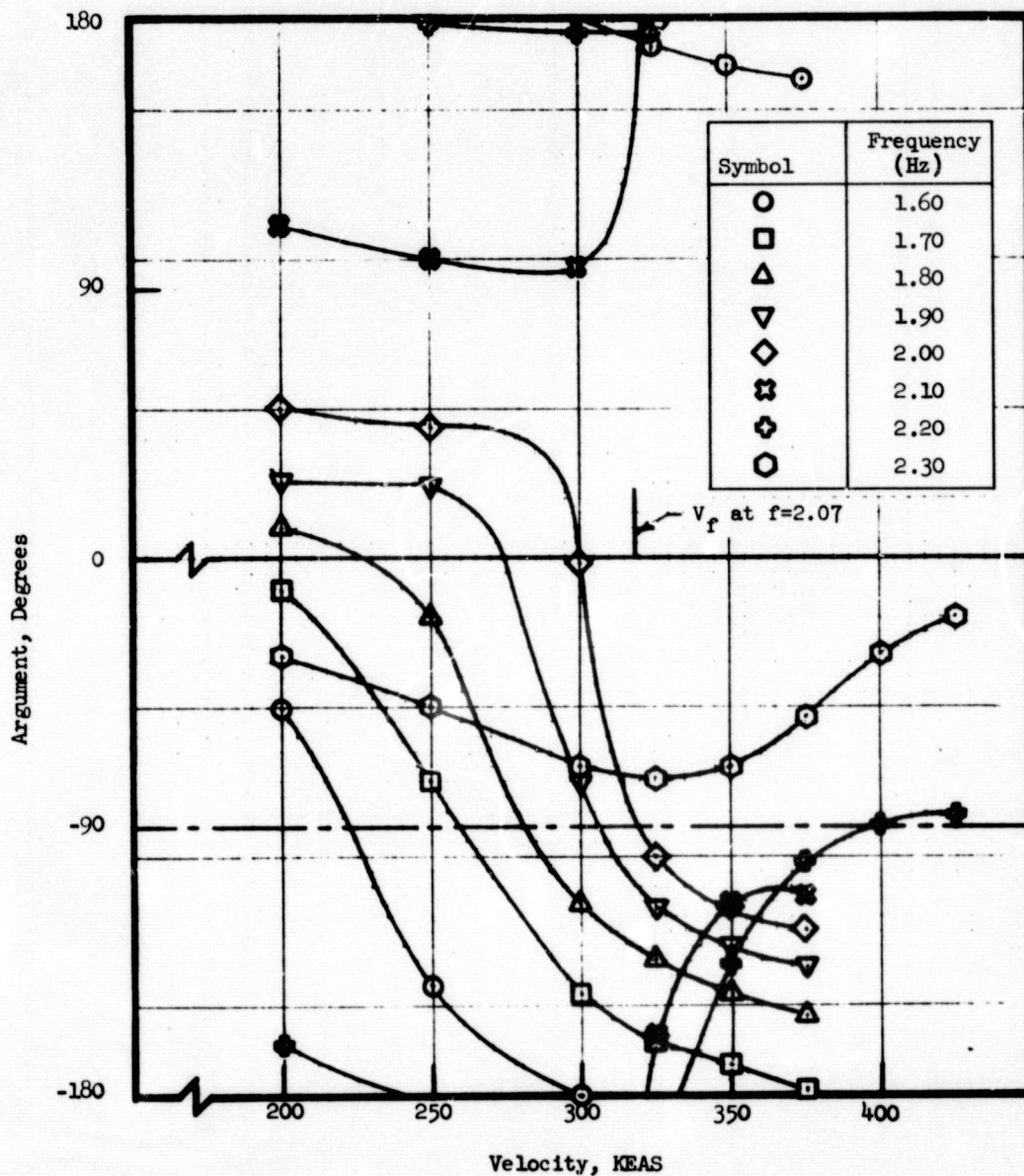


Figure 2-13: Case 1; Velocity Sweep at Several Frequencies;  $V_f$  is Flutter Velocity Obtained from Figure 2-6.



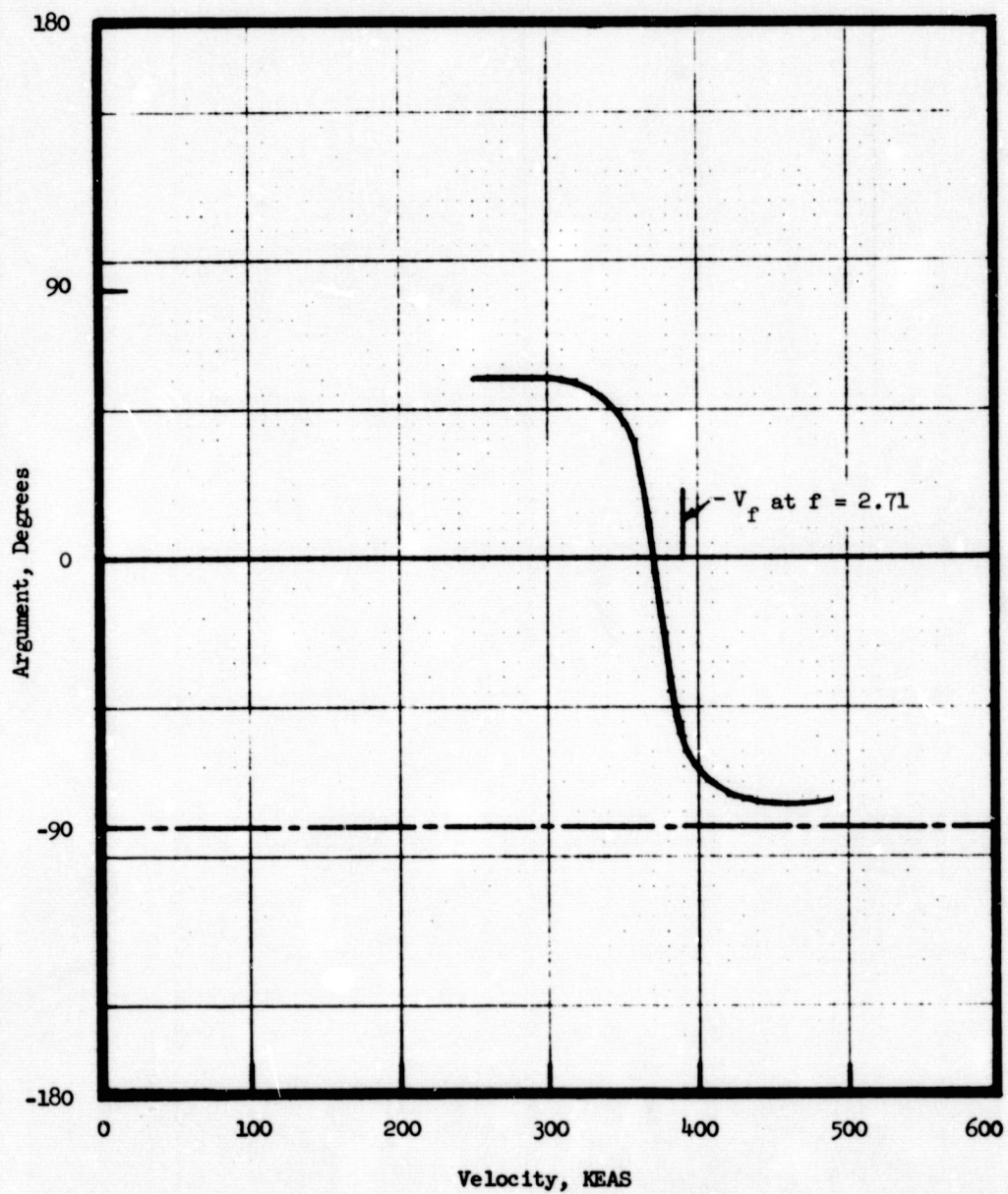


Figure 2-14: Case 2; Velocity Sweep for  $f = 2.65$  Hz;  $V_f$  is Flutter Velocity Obtained from Figure 2-8.



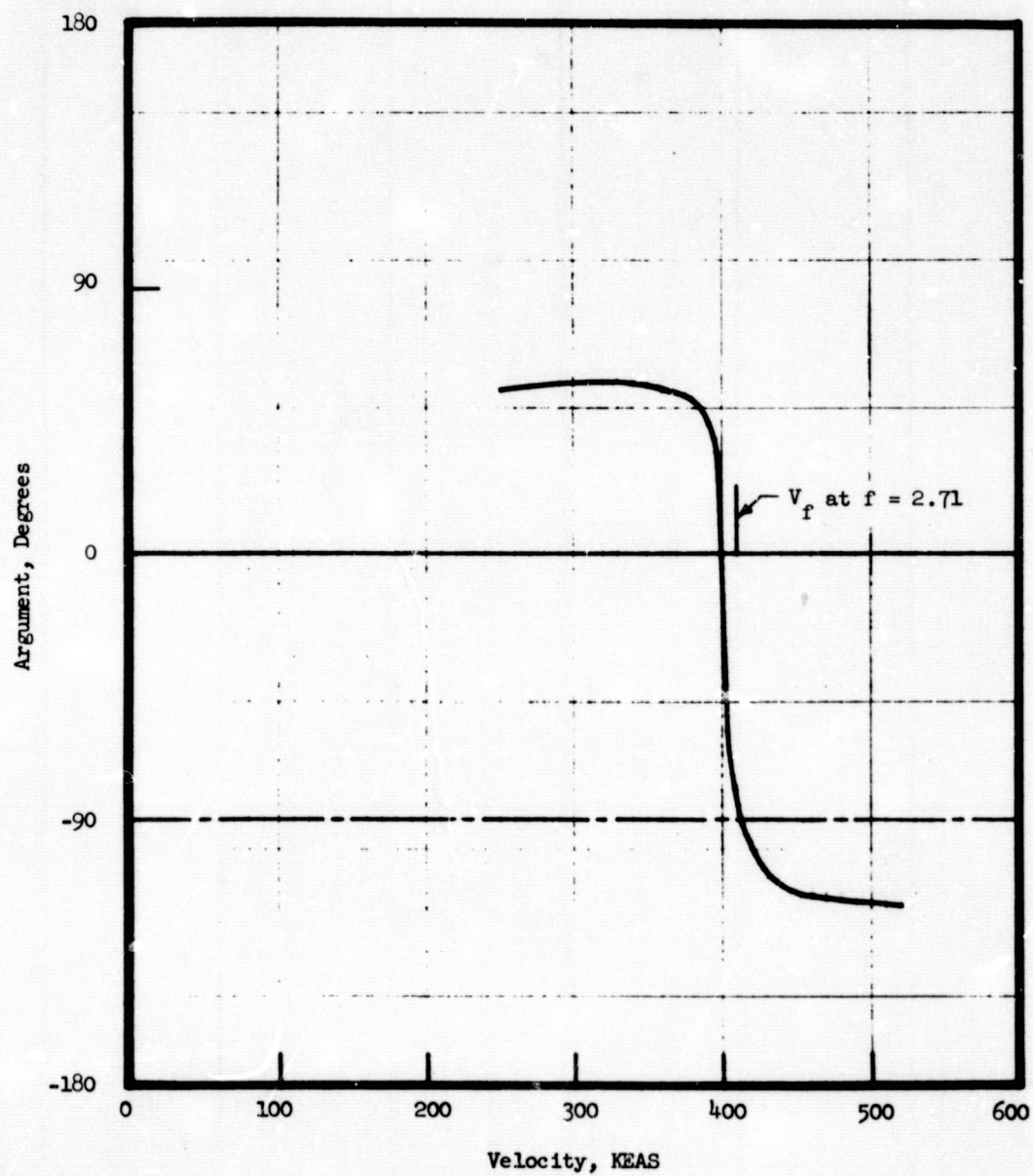


Figure 2-15: Case 3; Velocity Sweep for  $f = 2.70$  Hz;  $V_f$  is Flutter Velocity Obtained from Figure 2-11.

It is concluded that the velocity sweep is an uncertain method for a global search for flutter speeds. Possibly further investigations may lead to an effective approach.

2.3.3 Conclusion. The frequency sweep and velocity sweep, if made reliable, would provide a global search method for flutter speed and frequency that does not require the solution of characteristic value problems and that is free from the problems of mode following associated with the usual modal solutions. This potential gain justifies further examination of this approach.

#### 2.4 Description of the Basic Two-Dimensional Regula Falsi Program

Lockheed-California Company's two-dimensional Regula Falsi program solves the equation  $D(x,y) = 0$  for  $x$  and  $y$ , given a reasonable initial estimate of  $x$  and  $y$ .

The two equations implied by  $D(x,y) = 0$  are written as:

$$RD(x,y) = 0 \quad \text{and} \quad ID(x,y) = 0 \quad (2.15)$$

The two-dimensional Regula Falsi procedure is a separate sub-program which requires the values of  $RD(x,y)$  and  $ID(x,y)$  to be supplied through a calling list.  $RD(x,y)$  and  $ID(x,y)$  are therefore generated outside the Regula Falsi sub-program and can easily be adjusted to the problem at hand.

The iteration process is initiated by three pairs of values of  $x$  and  $y$  as described in Section 2.1. These values are defined as follows:

$$x_1 = x_0 ; \quad y_1 = y_0 \quad (2.16)$$

$$x_2 = x_1 + R_I \Delta x ; \quad y_2 = y_1 \quad (2.17)$$

$$x_3 = x_1 ; \quad y_3 = y_1 + R_I \Delta y \quad (2.18)$$

$x_0$  and  $y_0$  are estimated values of the solution, defined by the user. For the definition of the second and third trial pair the user must define  $R_I$ ,  $\Delta x$  and  $\Delta y$ .

Two modes of iteration are provided: Mode US (unrestricted stepsize) and Mode RS (restricted stepsize), to be defined later. A complete iteration may consist of one or more searches in Mode RS only, one search in Mode US, or combinations of Modes US and RS. If a combination is used, Mode US searches are conducted first, followed by the same number, or one less, of Mode RS searches. The user specifies the type of search desired and he

controls the number of searches by defining the maximum number of iterations per search and the maximum total number of iterations. Each time a new search is entered the iteration is reinitiated by defining new values of  $x_1, y_1$ , equal to the values  $x_m, y_m$  of the search just terminated. The quantities  $x_m$  and  $y_m$  are defined in connection with equation 2.21.

The quantities  $\Delta x$  and  $\Delta y$  define a search region. During the entire iteration the values of  $x$  and  $y$  are restricted to:

$$\begin{aligned} x_0 - \Delta x < x < x_0 + \Delta x \\ y_0 - \Delta y < y < y_0 + \Delta y \end{aligned} \quad (2.19)$$

This restriction is necessary to prevent the iteration process from being led away from the desired solution. If during the iteration either  $x$ , or  $y$ , or both exceed the boundary, the corresponding  $x$  or  $y$  that is in violation of the boundary is replaced according to a formula that depends on the mode of iteration, as discussed in a subsequent paragraph. Note that the boundaries are based on  $x_0$  and  $y_0$  and do not change when a new search is initiated.

The Type II Regula Falsi is used (Figure 2-2). Assume that the trials  $x_i, y_i, x_{i+1}, y_{i+1}, x_{i+2}, y_{i+2}$  have led to the next iterated value  $x_{i+3}, y_{i+3}$ . Then for the next iteration step  $x_{i+3}, y_{i+3}$  is used together with those values of  $x_j, y_j (j < i+3)$  that correspond to the smallest two values of the quantity  $T(x_j, y_j) = T_R(x_j, y_j) + T_I(x_j, y_j)$  which is defined by equation 2.20.

$$T_L(x_j, y_j) = \text{the larger of } \left| \frac{LD(x_j, y_j)}{LD(x_0, y_0)} \right| - 10^{-\epsilon} \text{ and } 0; L = R, I \quad (2.20)$$

where  $10^{-\epsilon}$  is a convergence criterion (equation 2.25).

For this purpose the program retains the values of  $RD$  and  $ID$  corresponding to the smallest two values of  $T(x_j, y_j)$ , and the associated  $x_j, y_j$  pairs as they occur during one search. Whenever a smaller  $T$  is encountered it replaces the larger of the two that were retained. When a new search is entered the  $x, y$  corresponding to the smallest  $T$  of the previous search become the  $x_0, y_0$  of the new search.

The distinction between the two modes of iteration,  $RS$  and  $US$ , lies in the stepsize. Stepsize is defined by

$$\delta x = |x_{i+3} - x_m|, \delta y = |y_{i+3} - y_m| \quad (2.21)$$

where  $x_m, y_m$  correspond to the smallest value of  $T$  during the current search.

In Mode US (unrestricted stepsize) there is no restriction on  $\delta x$  and  $\delta y$  except that equation 2.19 must be satisfied. In mode RS (restricted stepsize)  $\delta x$  and  $\delta y$  are restricted according to the equations:

$$\delta x \leq R_S R_I \Delta x \quad \delta y \leq R_S R_I \Delta y \quad (2.22)$$

The user defines the value of  $R_S$ .

When equations 2.19 and/or equations 2.22 are violated, corrective measures are taken. Several possibilities are recognized and treated as follows:

1. Mode RS and one of the equations 2.22 is violated; e.g.

$\delta x > R_S R_I \Delta x$ . Then  $x_{i+3}$  resulting from the last Regula Falsi step is replaced by  $\bar{x}_{i+3}$ .

$$\bar{x}_{i+3} = x_m + \frac{x_{i+3} - x_m}{|x_{i+3} - x_m|} \cdot R_S R_I \Delta x \quad (2.23)$$

Similarly for  $y$ .

2. Mode RS and one of the equations 2.19 is violated. The quantity in violation, say  $x_{i+3}$ , is replaced by  $\bar{x}_{i+3}$  according to equation 2.23. If  $\bar{x}_{i+3}$  still violates equations 2.19, the iteration is terminated.
3. Mode US and one of equations 2.19 is violated. The quantity in violation, say  $x_{i+3}$ , is replaced by  $\bar{\bar{x}}_{i+3}$

$$\bar{\bar{x}}_{i+3} = x_m + \frac{x_{i+3} - x_m}{|x_{i+3} - x_m|} 2R_I \Delta x \quad (2.24)$$

If  $\bar{\bar{x}}_{i+3}$  still violates equations 2.19, the iteration is terminated.

There are three further conditions under which the iteration is terminated:

1. A converged solution is obtained. The convergence criterion is:

$$\begin{aligned} RD_{\text{converged}} &\leq 10^{-\epsilon} RD(x_0, y_0) \quad \text{and} \\ ID_{\text{converged}} &\leq 10^{-\epsilon} ID(x_0, y_0) \end{aligned} \quad (2.25)$$

where  $\epsilon$  is defined by the user.

2. There are two consecutive occurrences, or three occurrences in total, of either  $x$  or  $y$  exceeding the boundaries defined by equation 2.19.
3. The number of iterations exceeds the total number of iterations permitted as specified by the user.

If the iteration is terminated without convergence either the entire calculation is terminated, or a recovery module is entered. At present there is one recovery module, called the Z-module, which is described in the next section.

## 2.5 The Z-Module

The Z-module provides a recovery procedure for the two-dimensional Regula Falsi program if the iteration is terminated without convergence having occurred. It is specifically designed to provide recovery under the circumstances encountered in an automatic resizing program with flutter constraints. It can be used, however, where the solution for a basic configuration is known and the solution for a modified solution is sought.

Let  $X$  stand for any combination of the variables  $x$  and  $y$ . Let  $D_B(X) = 0$  be the determinantal equation for the base configuration, and  $D_B(X) + \Delta D(X) = 0$  the equation for the modified configuration;  $X_B$  is the known solution of  $D_B(X) = 0$  and  $X_0$  is the estimated solution of  $D_B(X) + \Delta D(X) = 0$ .

Assume that solving  $D_B(X) + \Delta D(X) = 0$  with  $X_0$  initiating the iteration led to a termination without convergence. The Z-module then generates

a new determinantal equation  $DB(X) + Z \Delta D(X) = 0$  and first trial value  $X = X_B + Z(X_0 - X_B)$ .  $Z$  is a fraction smaller than unity and in effect the  $Z$ -module reduces the difference between the modified and the base configuration. If  $X_0$  is chosen equal to  $X_B$ , a convenient choice in flutter optimization,  $X_0 = X_B$  tends to be a better first trial value for  $D_B(X) + Z\Delta D(X) = 0$  than for  $D_B(X) + \Delta D(X) = 0$ .

The first value of  $Z$  is  $Z = 0.5$ . If this does not lead to convergence  $Z = 0.25$  is used. In general  $Z = (0.5)^n$ , where  $n$  is the number of successive occurrences of non-convergence. The value of  $Z$  at which convergence occurs is identified as  $Z_C$  and the corresponding solution as  $X_C$ . New trial values for the original problem  $D_B(X) + \Delta D(X) = 0$  are obtained by linear extrapolation using  $X_B$  and  $X_C$ .

$$X_0 = X_B + \frac{X_C - X_B}{Z_C} \quad (2.26)$$

If non-convergence occurs again, the design condition at  $Z_C$  becomes the new base design:

$$\overline{D}_B(X) = D_B(X) + Z_C \Delta D(X) \quad (2.27)$$

$$\text{and } \overline{\Delta D}(X) = (1 - Z_C) \Delta D \quad (2.28)$$

and  $n$  is set at  $n = 1$  and  $Z$  is reset to 0.5.

The reduction procedure is repeated until another value of  $Z$  at which convergence occurs is found. At this point the three known solutions are used in obtaining new trial values for  $Z = 1.0$ .

The above iterative process is continued until either convergence is obtained at  $Z = 1.0$  or until  $n = n_{MAX}$ , where

$n_{MAX}$  = maximum allowable successive convergence failures.

## 2.6 Numerical Examples

Many numerical examples have been used to develop and test the two-dimensional Regula Falsi Program and the Z-module. The example in table 2-1 is chosen to illustrate the program. The final solution can be seen to lie outside the search region. The initial trial is made equal to the base solution. The first search is terminated because of excessive violations of the search limits. The Z module sets  $Z = 0.5$ , cutting the difference between the base problem and the new problem in half. When again there are more than the allowed number of search limit violations  $Z$  is reduced to  $0.25$  for which value convergence occurs. This solution and the base solution are used to generate an improved initial trial value for the new problem ( $Z=1$ ) which leads to a converged solution.

## 3. INCREMENTED FLUTTER ANALYSIS

Incremented Flutter Analysis is a useful tool that can enhance several methods of optimization with flutter constraints (Reference 1). It is a method for directly solving for the value of a design variable that satisfies a given flutter constraint. In the original presentation of the method (Reference 4) this basic idea was coupled with a formulation that led to a large reduction of the order of the characteristic value problem to be solved without losing any of the accuracy implied by the original order. This formulation is useful for problems with a small number of design variables, such as the determination of external store parameters satisfying the flutter speed requirements. When the number of design variables is large, and when they can not easily be isolated in the mathematical representation of the problem (e.g., if problem is modalized) the reduction of the order of the characteristic problem to be solved becomes impracticable. Such is often the case with flutter optimization. In this section the formulation of Incremented Flutter Analysis for general application in flutter optimization programs is presented, together with applications.

It should be noted that the essence of Incremented Flutter Analysis was recognized elsewhere (Reference 5).

### 3.1 Form of the Flutter Equation

The characteristic flutter equation can be written as

$$D(g, \gamma, k, V, m_i) = 0 \quad (3.1)$$

INPUT: Allowable Iterations per search = 7, Allowable Iterations per Z = 10  
 Allowable Iterations per run = 40,  $x_0 = 2.89$ ,  $y_0 = 453$ ,  $\Delta x = .2$ ,  $\Delta y = 20$

$$R_I = .2, R_S = .5, \epsilon = 10^{-3}, x_B = 2.89, y_B = 453$$

RESULTS:

Iteration Number	x	y	RD	ID	Z	COMMENTS
1	2.890	453.00	-0.8585	1.1415	1.00	Initial Trials: $x_1 = x_0, y_1 = y_0$
2	2.930	453.00	-1.1542	1.3529		$x_2 = x_0 + R_I \Delta x, y_2 = y_0$
3	2.890	457.00	-0.8343	1.0854		$x_3 = x_0, y_3 = y_0 + R_I \Delta y$
4	2.818	465.00	-0.5746	0.5006		Upper y search limit violated.
5	2.738	457.00	-0.4335	0.1322		Lower x and y search limits violated. (Discontinued due to excessive search limit violations)
6	2.890	453.00	-0.8852	1.1148	0.50	Initial trials are the same as at Z = 1.0 since $x_0 = x_B$ and $y_0 = y_B$  Lower x and y search limits violated Upper y search limit violated. (Discontinued due to excessive search limit violations)
7	2.930	453.00	-1.0788	1.4557		
8	2.890	457.00	-0.8186	0.9703		
9	2.810	449.00	-0.7219	0.4141		
10	3.076	457.00	-1.9066	0.8490		

Table 2-1: Numerical Example of Two-Dimensional Regula Falsi Iteration Including the Use of the Z Module



# RESULTS

Iteration Number	x	y	RD	ID	Z	COMMENTS
11	2.890	453.00	-0.8511	1.1489	0.25	Initial trials are the same as at Z = 1.0 since $y_0 = x_B$ and $y_0 = y_B$  Upper y search limit violated.  Converged for Z = 0.25.
12	2.930	453.00	-0.8566	1.7888		
13	2.890	457.00	-0.6953	0.8338		
14	2.926	465.00	-0.0107	0.7615		
15	2.906	470.67	0.0958	-0.1110		
16	2.904	469.09	-0.0197	0.0128		
17	2.905	469.32	0.0007	0.007		
18	2.949	518.27	-1.2027	-0.7973	1.00	x and y obtained by linear extra- polation using $x_B, y_B$ and $x_{17}, y_{17}$  Converged at Z = 1.00
19	2.989	518.27	1.5326	0.4762		
20	2.949	522.27	-0.5506	-2.0998		
21	2.968	517.68	-0.0542	-0.0586		
22	2.968	517.84	-0.0055	0.0171		
23	2.968	517.88	0.00008	0.0004		

Table 2-1: Numerical Example of Two-Dimensional Regula Falsi Iteration Including the Use of the Z Module (Continued)

The quantity  $D$  is called the flutter determinant and has a real and imaginary part. Thus equation 3.1 represents two equations. For this discussion  $D$  is considered a function of the structural damping  $g$ , the decay rate  $\gamma$ , the reduced frequency  $k$ , the velocity  $V$  and the design variables  $m_i$  ( $i=1 \rightarrow n$ ).

Since equation 3.1 represents two equations it can be solved for two unknowns. In the traditional  $k$  method of solving the flutter equation  $\gamma = 0$ ,  $m_i$  is fixed and for a series of values of  $k$  equation 3.1 is solved

for  $g$  and  $V$ . In the  $p$ - $k$  method of solving the flutter equation  $g$  and  $m_i$  are fixed, and for a series of values of  $V$  the equation is solved for  $\gamma$  and  $k$ .

Letting  $\gamma = 0$  and giving  $g$  and  $m_i$  fixed values, equation 3.1 can be used to solve directly for the flutter speed and the associated reduced frequency. This use of the flutter equation has gained importance in connection with structural optimization with flutter constraints.

Incremented Flutter Analysis is characterized by the addition of  $m_i$  to the variables of which  $D$  is a function. Thus the meaning and use of the flutter equation is generalized. When in equation 3.1 the values of  $g$ ,  $\gamma$ ,  $V$  and all but one of the  $m_i$ 's are fixed it can be solved for the value of  $k$  and the variable  $m_i$ . If a real solution exists, the value of the unknown  $m_i$  is found that, together with the fixed  $m_i$ 's defines a structure that at the given speed and structural damping exhibits the given rate of decay. If  $\gamma = 0$ , the given  $V$  is the flutter speed, and thus a structure has been defined with a preset flutter speed.

### 3.2 Application to Finite-Difference Technique

Most methods of structural optimization with flutter constraints require the evaluation of the derivatives of the flutter properties with respect to the design variables. Analytic expressions for the flutter-speed derivative (Reference 6) and the derivative of the logarithmic increment (Section 7.2) have been derived and applied successfully. However, derivatives obtained by the finite-difference technique have also been used successfully (Reference 7). This section shows how the forward finite difference quotient of the flutter speed with respect to a design variable can be determined with the help of the Incremented Flutter Analysis Technique.

Let  $\{m_i\}$  define a structure with a known flutter speed  $V$ . Define a perturbed structure by  $\{\bar{m}_i\}$ , where  $\{\bar{m}_i\}$  is obtained from  $\{m_i\}$  by increasing the single  $j^{\text{th}}$  element by  $\Delta m_j$ . Assume this change increases the flutter speed an amount  $\Delta V$ .

Write the flutter equation as:

$$D(g, \gamma, k, V, \{\bar{m}_i\}, \Delta m_i) = 0 \quad (3.2)$$

Let  $\gamma = 0$ ,  $g$  = structural damping,  $V$  the original flutter speed and  $\{\bar{m}_i\}$  constant. Solve for each  $\Delta m_i$  ( $i = 1 \rightarrow n$ ) separately, and the associated  $k$  value (which is of interest only for checking purposes). Each  $\Delta m_i$  found is the amount that must be added to the  $i^{\text{th}}$  design variable in the perturbed system such that each  $\Delta m_i$  by itself restores the flutter speed to its original value. By definition

$$(\Delta m_i)_{i=j} = -\Delta m_j \quad (3.3)$$

The values of  $\Delta m_i$  thus determined can be used to approximate derivatives of the flutter speed with respect to the design variables.

$$\frac{\partial V}{\partial m_i} \approx -\frac{\Delta V}{\Delta m_i} \quad (3.4)$$

This requires the numerical evaluation of  $\Delta V$ . For certain procedures, only normalized flutter speed derivatives are needed. Let the subscript  $R$  refer to a reference design variable. Then:

$$\left( \frac{\partial V}{\partial m_i} \right)_{\text{norm}} = \frac{\frac{\partial V}{\partial m_i}}{\frac{\partial V}{\partial m_R}} \approx \frac{\Delta m_R}{\Delta m_i} \quad (3.5)$$

and there is no need to compute  $\Delta V$ .

### 3.3 Determining the Magnitude of a Resizing Column

When raising the flutter speed from an initial, deficient value to the required value, and in several methods of structural optimization with flutter constraints, it is necessary to determine the magnitude of a given design variable distribution column such that the flutter equation is satisfied at a given flutter speed.

Incremented Flutter Analysis provides a technique for determining this magnitude directly, without need of generating several solutions to be used in an interpolation procedure.

The statement of the problem is: given a structure with an unknown flutter speed, defined by  $\{m_i\}$ , and a distribution of design variable increments  $\{\Delta m_i\}$ , determine a scalar  $C$  such that the structure defined by  $\{m_i\} + C \{\Delta m_i\}$  has a given flutter speed  $V$ .

The solution is found by writing equation 3.1 as:

$$D(g, \gamma, k, V, \{m_i\}, C\{\Delta m_i\}) = 0 \quad (3.6)$$

and solving for  $C$  and  $k$ .

An extension of this procedure can be used to determine the relative magnitude of two resizing columns as might be needed in a one-dimensional minimization routine such as the one described in Section 6.6 of Reference 1. For this purpose equation 3.6 is written in the form:

$$D(g, \gamma, k, V, \{m_i\}, \bar{C} \{\Delta \bar{m}_i\}, C \{\Delta m_i\}) = 0 \quad (3.7)$$

Let  $\gamma = 0$ , and  $g, V, \{m_i\}, \{\Delta \bar{m}_i\}$  and  $\{\Delta m_i\}$  be fixed. For successive values of  $\bar{C}$  the corresponding values of  $C$  can be determined such that the flutter speed has the desired value. The one-dimensional minimization then consists of determining the value of  $\bar{C}$  for which  $\Delta M = \bar{C} \Sigma \Delta \bar{m}_i + C \Sigma \Delta m_i$  is minimum.

### 3.4 Concluding Remarks

The finite difference technique using Incremented Flutter Analysis has been used successfully for approximating flutter speed derivatives in the numerical evaluation performed for the method of optimization described in Section 6.6 of Reference 1. It was concluded, however, that as the number of design variables with respect to which the derivatives need to be determined increases, the analytical approach to determining flutter speed derivatives (Reference 6) becomes relatively more efficient. Nevertheless, the technique described may be useful in special cases.

Determining the magnitude of a column of design variable increments such that a flutter speed is exactly satisfied can be a useful tool in all methods of optimization discussed in Reference 1, except the penalty function method. It eliminates the need for more than one solution of the characteristic flutter equation and subsequent interpolation.

## 4. AERODYNAMICS

One of the objectives of this study is to determine general, efficient and accurate computational forms in which to cast the unsteady aerodynamic parameters necessary for use in structural resizing required to meet flutter constraints. This effort, however, did not involve evaluation of aerodynamic theories. A general discussion and conclusions are presented in Reference 1. Additional discussion is presented in this Section.

Reference 1 indicates that an aerodynamic theory based on the use of the aerodynamic velocity potential leads to the same basic five matrix product for the generalized aerodynamic force coefficients as the kernel function approach of Reference 8. Section 4.1 presents more detail.

The problem of "hunting" when using aerodynamic matrices, interpolated for arbitrary values of  $k = \frac{\omega C}{V}$ , may occur if the interpolation is piecewise, i.e., each interpolation covers only part of the range of  $k$  values. This problem is mentioned in Reference 1 as related to jumps in the derivative when the third degree polynomial interpolation formula is simply differentiated. Section 4.2 discusses a form of "hunting" which can occur even if no derivatives are used.

As part of the present study an effort was made to come to definitive conclusions regarding details of an efficient procedure for computing the generalized aerodynamic force coefficients for use in the repetitive calculations required in a structural weight minimization procedure. Formulas for the number of computational operations needed to generate the aerodynamics matrix have been derived and sample calculations have been made. This work is summarized in Section 4.3.

### 4.1 The Use of Velocity Potential Influence Coefficients

A generally useful formulation of the unsteady aerodynamic forces is one in which a matrix of aerodynamic influence coefficients,  $[AIC(k)]$ , expresses lumped aerodynamic loads at an aerodynamic loads grid in terms of angles of attack at downwash collocation points:

$$\{Z\} = [AIC(k)] \{\alpha\} \quad (4.1)$$

Such a matrix can be obtained regardless of which aerodynamic theory is used.

Two types of theoretical approach are widely used: one is based on the use of the acceleration potential, the other on the use of the velocity potential. Three methods of computing unsteady aerodynamic forces are

commonly used: subsonically, the integral kernel formulation of Reference 8 and the finite element approach of Reference 9, both based on the use of the acceleration potential; supersonically, the supersonic mach box approach of Reference 10, which is based on the use of the velocity potential.

Recently the velocity potential approach has received increased attention as a common basis for the computation of subsonic, transonic and supersonic aerodynamics (References 11 through 15), and the question arose whether a formulation in terms of velocity potential influence coefficients, VPIC, might have computational advantages when used in an optimization procedure. Therefore in the following, theoretical expressions are developed to express the generalized aerodynamic force coefficients in terms of the five matrix product of Reference 1, assuming velocity potential influence coefficients are available.

4.1.1 Basic equations. The generalized aerodynamic force is defined by

$$A_{ij} = \iint p_j h_i dx dy \quad (4.2)$$

where  $p_j$  is the lifting pressure distribution associated with the  $j^{\text{th}}$  mode and  $h_i$  the displacement associated with the  $i^{\text{th}}$  mode.

The pressure distribution can be expressed in terms of the velocity potential  $\phi$ . The linearized Bernoulli equation gives:

$$p = -2 \left( \frac{\partial \phi}{\partial x} + ik\phi \right) \quad (4.3)$$

where  $k$  is the reduced frequency:  $k = \frac{\omega c}{V}$ .

Thus equation 4.2 becomes:

$$A_{ij} = -2 \iint \left( \frac{\partial \phi_j}{\partial x} + ik\phi_j \right) h_i dx dy \quad (4.4)$$

Direct integration of equation 4.4 gives:

$$A_{ij} = -2 \int_{-b/2}^{b/2} \left\{ \int_{L.E.}^{T.E.} ik\phi_j h_i dx + \int_{L.E.}^{T.E.} h_i \frac{\partial \phi_j}{\partial x} dx \right\} dy \quad (4.5)$$

Integrating the term with  $\frac{\partial \phi}{\partial x}$  by parts gives:

$$A_{ij} = -2 \int_{-b/2}^{b/2} \int_{L.E.}^{T.E.} ik\phi_j h_i dx + (h_i \phi_j)_{T.E.} - \int_{L.E.}^{T.E.} \phi_j \frac{\partial h_i}{\partial x} dx dy \quad (4.6)$$

In performing the integrations in equations 4.5 and 4.6 by numerical integration one may consider first doing the chordwise integration at several stations  $y = \text{constant}$  along the span, and then doing the spanwise integration. In following this procedure it is found that inconvenient matrix formulations result in which mode-dependent and mode-independent parts are intermixed. Therefore the numerical integration is formulated as an area integration. Thus equation 4.5 is written as:

$$A_{ij} = -2 \int_{-b/2}^{b/2} \int_{L.E.}^{T.E.} ik\phi_j h_i dx dy - 2 \int_{-b/2}^{b/2} \int_{L.E.}^{T.E.} h_i \frac{\partial \phi_j}{\partial x} dx dy \quad (4.7)$$

and equation 4.6 as:

$$A_{ij} = -2 \int_{-b/2}^{b/2} \int_{L.E.}^{T.E.} ik\phi_i h_i dx dy + 2 \int_{-b/2}^{b/2} \int_{L.E.}^{T.E.} \phi_j \frac{\partial h_i}{\partial x} dx dy - 2 \int_{-b/2}^{b/2} (h_i \phi_j)_{T.E.} dy \quad (4.8)$$

Equation 4.8 has one more term than equation 4.7 but has the advantage that the integrand does not become infinite at the leading edge.

4.1.2 The two-term integration. Written in a form for numerical integration equation 4.7 becomes:

$$A_{ij} = -2ik \begin{bmatrix} w_1 \end{bmatrix} \begin{bmatrix} h_i \end{bmatrix} \left\{ \phi_j \right\} - 2 \begin{bmatrix} w_2 \end{bmatrix} \begin{bmatrix} h_i \end{bmatrix} \left\{ \frac{\partial \phi_j}{\partial x} \right\} \quad (4.9)$$

where  $[W_1]$  and  $[W_2]$  are integrating matrices associated with integration points at which  $h_i$  is defined. The option that  $[W_1] \neq [W_2]$  is included to provide the possibility of adjusting the integrating matrices to the shape of the functions to be integrated.

Let  $h_i$  be defined at  $y=\text{constant}$  stations at which  $\phi_i$  is defined, but assume the general case that  $h_i$  and  $\delta_i$  are defined at different points in the  $x$  direction.

Let  $[IPHX]$  be an interpolating matrix and  $[DPHX]$  a differentiation matrix such  $\phi_j$  and  $\frac{\partial \phi_j}{\partial x}$  at the integration points are defined, respectively, by

$$[IPHX] \{\phi_j\} \text{ and } [DPHX] \{\phi_j\} \quad (4.10)$$

Substituting the expressions 4.10 in equation 4.9 and interchanging the  $W$  and  $h$  matrices gives:

$$A_{ij} = [h_i] \left[ -2ik [W_1] [IPHX] - 2 [W_2] [DPHX] \right] \{\phi_j\} \quad (4.11)$$

which, with  $-2 [W_1] [IPHX] = [WF]$  and  $-2 [W_2] [DPHX] = [WFD]$ , corresponds to equation 5.11 of Reference 1, generalized to non-coinciding  $h_i$  and  $\phi_i$

stations. The velocity potential can be expressed as the product of a matrix of velocity potential influence coefficients and an angle-of-attack distribution.

$$\{\phi_j\} = [VPIC] \{\alpha_j\} \quad (4.12)$$

The angle-of-attack distribution can be related to the structural displacements,  $\{z\}$ , by a differentiating matrix  $[DX]$ , generating  $\frac{\partial z}{\partial x}$ , and an interpolating matrix  $ik[DZ]$ , generating the frequency dependent equivalent of  $\frac{\dot{z}}{V}$ :

$$\{\alpha_j\} = \left[ [DX] + ik [DZ] \right] \{z_j\} \quad (4.13)$$



Combining the equations 4.11, 4.12 and 4.13:

$$A_{ij} = [h_i] \left[ -2ik [W_1] [IPHX] - 2 [W_2] [DPHX] \right] [VPIC] \left[ [DX] + ik [DZ] \right] \{z_j\} \quad (4.14)$$

$$\text{Let } [AIC] = \left[ -2ik [W_1] [IPHX] - 2 [W_2] [DPHX] \right] [VPIC] \quad (4.15)$$

$$[W] = [DX] + ik [DZ] \quad (4.16)$$

$[h_i]$  is obtained from the structural displacement  $[z_i]$  by interpolation:

$$[h_i] = [z_i] [H]^T \quad (4.17)$$

With equations 4.15, 4.16 and 4.17 equation 4.14 becomes:

$$A_{ij} = [z_i] [H]^T [AIC] [W] \{z_j\} \quad (4.18)$$

For several modes  $z_i$  and  $z_j$  this becomes

$$[A_{ij}] = [\bar{z}]^T [H]^T [AIC] [W] [\bar{z}] \quad (4.19)$$

This equation is identical to equation 5.12 in Reference 1 and the [AIC] matrix can be evaluated by any suitable method.

4.1.3 The three-term integration. Write equation 4.8 in a form for numerical integration:

$$A_{ij} = -2ik [W_1] [h_i] \{\phi_j\} + 2 [W_3] \left[ \frac{\partial h_i}{\partial x} \right] \{\phi_j\} + \quad (4.20)$$

$$- 2 [W_4] [h_{i_{T.E.}}] \{\phi_{j_{T.E.}}\}$$

After interchanging the W and h matrices equation 4.20 becomes

$$A_{ij} = -2ik \begin{bmatrix} h_i \end{bmatrix} \begin{bmatrix} W_1 \end{bmatrix} \{\phi_j\} + 2 \begin{bmatrix} \frac{\partial h_i}{\partial x} \end{bmatrix} \begin{bmatrix} W_3 \end{bmatrix} \{\phi_j\} +$$

$$- 2 \begin{bmatrix} h_{i_{T.E.}} \end{bmatrix} \begin{bmatrix} W_4 \end{bmatrix} \{\phi_{j_{T.E.}}\}$$
(4.21)

As before  $\{\phi_j\}$  is replaced by  $[IPHX] \{\phi_j\}$ .

Similarly:

$$\{\phi_{j_{T.E.}}\} = [EP] \{\phi_j\},$$
(4.22)

where  $[EP]$  extrapolates  $\{\phi_j\}$  to the trailing edge.

$\begin{bmatrix} \frac{\partial h_i}{\partial x} \end{bmatrix}$  is formed from  $\begin{bmatrix} h_i \end{bmatrix}$  with the help of a differentiating matrix  $[DHX]$ :

$$\begin{bmatrix} \frac{\partial h_i}{\partial x} \end{bmatrix} = \begin{bmatrix} h_i \end{bmatrix} [DHX]^T$$
(4.23)

and  $\begin{bmatrix} h_{i_{T.E.}} \end{bmatrix}$  is found by extrapolation similar to equation 4.22.

$$\begin{bmatrix} h_{i_{T.E.}} \end{bmatrix} = \begin{bmatrix} h_i \end{bmatrix} [EH]^T$$
(4.24)

Combining equations 4.21 through 4.24 the expression

$$A_{ij} = \begin{bmatrix} h_i \end{bmatrix} \left[ -2ik \begin{bmatrix} W_1 \end{bmatrix} [IPHX] + 2 [DHX]^T \begin{bmatrix} W_3 \end{bmatrix} [IPHX] + \right.$$

$$\left. - 2 [EH]^T \begin{bmatrix} W_4 \end{bmatrix} [EP] \right] \{\phi_j\}$$
(4.25)

is obtained.

Letting

$$\begin{aligned}
 [AIC] = & \left[ -2ik \begin{bmatrix} W_1 \end{bmatrix} [IPHX] + 2 [DHX]^T \begin{bmatrix} W_3 \end{bmatrix} [IPHX] + \right. \\
 & \left. - 2 [EM]^T \begin{bmatrix} W_4 \end{bmatrix} [EP] \right] [VPIC]
 \end{aligned}
 \tag{4.26}$$

and following the procedure that led to equation 4.19, again an equation is obtained that can be written as

$$[A_{ij}] = [\bar{z}]^T [H]^T [AIC] [W] [\bar{z}]
 \tag{4.27}$$

4.1.4 Conclusion. It has been shown that rather simple matrix equations relate the generalized aerodynamic forces to velocity potential influence coefficients. Whether equation 4.7 is followed or equation 4.8, the same expression for the matrix of generalized aerodynamic influence coefficients is found when using aerodynamics based on the velocity potential as is found when using the integral kernel function approach (Equation 5.12 of Reference 1).

## 4.2 Hunting Due to Piecewise Interpolation

It is generally accepted that when the matrix of generalized aerodynamic force coefficients,  $[A(k)]$ , is determined for a discrete set of values,  $k_\ell$ , of the reduced frequency, interpolation is adequate for approximating  $[A(k)]$  at arbitrary values of  $k$ . One method of interpolation, used successfully at the Lockheed California Company, is the cubic polynomial.

For the cubic polynomial Lagrange's interpolation formula is considered to be most efficient since it expresses  $[A(k)]$  directly in terms of its value  $[A(k_\ell)]$ , at discrete values  $k_\ell$ ,  $\ell = 1, 2, 3, 4$ :

$$[A(k)] = \sum_{\ell=1}^4 \mathcal{L}_\ell(k) [A(k_\ell)]
 \tag{4.28}$$

where  $\mathcal{L}_\ell(k)$  is defined such that:

$$\mathcal{L}_1(k) = \frac{(k-k_2)(k-k_3)(k-k_4)}{(k_1-k_2)(k_1-k_3)(k_1-k_4)} \quad (4.29)$$

Cyclic substitution leads to  $\mathcal{L}_2$ ,  $\mathcal{L}_3$  and  $\mathcal{L}_4$ .

The interpolation formula 4.28 is considered most accurate for the interval  $k_2 < k < k_3$ . For the interval  $k_3 < k < k_4$  the index  $\ell$  must be increased by one; for  $k_1 < k < k_2$  the index must be lowered by one.

Reference 1 indicates that direct differentiation of equation 4.28 leads to jumps in the derivative of  $[A(k)]$  at all values  $k = k_\ell$  if the cubic polynomial formula is reindexed as  $k$  moves from interval to interval. The jumps in the derivative can be avoided if the differentiation precedes the interpolation. This is accomplished by evaluating  $[A'(k)]$  at  $k_\ell$  and determining  $[A'(k)]$  for arbitrary  $k$  from  $[A'(k_\ell)]$  in the same way as  $[A(k)]$  is determined from  $[A(k_\ell)]$ . These jumps in the derivatives of the aerodynamics matrix could lead to oscillatory non-convergence ("hunting") of the structural resizing if the resizing procedure used requires the derivatives of the flutter speed. The problem does not occur if one interpolation formula is used for the entire  $k$  range or if special provisions for continuity are made at the initial set of discrete  $k$  values:  $k_\ell$ . The latter is the case if cubic spline interpolation is used.

A possibility of hunting exists that is not related to jumps in the value of the derivative. It has occurred while solving the flutter equation according to the  $p$ - $k$  method (Reference 3). In the  $p$ - $k$  method a modified one-dimensional Regula Falsi is used to solve for the stability roots  $p = (\gamma + i)k$  at constant velocity (see also Section 2.2). In the Lockheed-California Company's  $p$ - $k$  method computer program, cubic polynomial interpolation is used for the aerodynamic matrix. As the value of  $k$  moves from one interval to the next during the iterative solution process, two different aerodynamic polynomials are actually used in solving the  $p$ . This can lead to hunting, in which the solution is sent back and forth between two adjacent  $k$  intervals. This type of hunting has been overcome by allowing  $k$  to take values that lie some distance into an adjacent interval without changing the interpolation formula. Thus the interpolation formula for the  $k_2$ - $k_3$  interval

is actually used if  $k_2 - \epsilon < k < k_3 + \epsilon$ . Only when  $k$  moves out of this modified interval is the interpolation formula for the modified adjacent interval used. Thus for a  $k$  range equal to  $2\epsilon$ , straddling each  $k_l$ , the aerodynamics matrix is not uniquely defined. Actual experience with the modified interval is restricted to  $\epsilon$  equal to half the adjacent interval. This has not led to practical difficulties.

#### 4.3 Comparison of Options for Forming the Aerodynamics Matrix

The matrix of generalized aerodynamic force coefficients was formulated in equation 4.19 as:

$$[A(k)] = [\bar{z}]^T [H]^T [AIC(k)] [W(k)] [\bar{z}] \quad (4.31)$$

where  $[AIC(k)]$ , a function of the reduced frequency  $k = \frac{\omega c}{V}$  and the Mach number, is the core of the aerodynamics. It is independent of mode shape and can be evaluated by any suitable aerodynamics method. Its elements are basic aerodynamic influence coefficients defining lumped aerodynamic forces  $\{Z_a\}$  at an aerodynamic force grid in terms of the angles of attack at downwash collocation points:

$$\{Z_a\} = [AIC(k)] \{\alpha\}.$$

The matrix  $[W(k)] = \left[ [DX] + ik[DZ] \right]$  relates the angles of attack to the structural displacements  $\{z\}$ . It is independent of mode shape.

The matrix  $[H]^T$  is independent of  $k$  and of mode shape, and distributes lumped aerodynamic forces and moments to the structural grid.

The matrix  $[\bar{z}]$  is the matrix of modal columns in terms of the structural displacements  $\{z\}$ .

The matrices  $[AIC(k)]$ ,  $[H]^T$  and  $[W(k)]$  are constant during an optimization procedure. They will be used many times during the design process of an airplane with a given external configuration. It is therefore advantageous to form these matrices in a special aerodynamics computer program.

Each time during an optimization procedure that a remodalization takes place  $[A(k)]$  must be recomputed. Depending on the dimensions of the matrices in equation 4.31 it may be more efficient to compute the triple matrix product

$$[HAW(k)] = [H]^T [AIC(k)] [W(k)] \quad (4.33)$$

in the aerodynamics program, or to perform one or both of the multiplications  $[\bar{z}]^T [H]^T$  and  $[W(k)][\bar{z}]$  in the optimization program.

In this section the possible sequences of multiplication in equation 4.31 in the form given, and with  $[W(k)]$  replaced by  $[DX] + ik [DZ]$ , are examined and compared. They are identified as Options H1a, H1b, H2a, H2b, H3, H4a, H4b and H5 and they are defined in Section 4.3.2. In all options it is assumed that the derivative of the aerodynamics matrix with respect to  $k$  is needed.

Since the aerodynamics matrix and its derivative need to be evaluated at arbitrary values of  $k$ , interpolation is necessary. Two interpolation options are considered: cubic polynomial and cubic spline.

For cubic polynomial interpolation the value of the aerodynamics matrix  $[A(k_\ell)]$  and its derivative  $[A'(k_\ell)]$ , evaluated at discrete values  $k = k_\ell$  are used in a Lagrangian interpolation formula:

$$[A(k)] = \sum_{\ell=1}^4 \mathcal{L}_\ell(k) [A(k_\ell)] \quad (4.34)$$

and

$$[A'(k)] = \sum_{\ell=1}^4 \mathcal{L}_\ell(k) [A'(k_\ell)] \quad (4.35)$$

where:

$$\mathcal{L}_1(k) = \frac{(k-k_2)(k-k_3)(k-k_4)}{(k_1-k_2)(k_1-k_3)(k_1-k_4)} \quad (4.36)$$

and cyclic substitution leads to  $\mathcal{L}_2$ ,  $\mathcal{L}_3$  and  $\mathcal{L}_4$ .

For cubic spline interpolation the expressions for  $[A(k)]$  and  $[A'(k)]$  are:

$$[A(k)] = [\bar{A}_0] + [\bar{A}_1] (k-k_\ell) + [\bar{A}_2] (k-k_\ell)^2 + [\bar{A}_3] (k-k_\ell)^3 \quad (4.37)$$

and

$$[A'(k)] = [\bar{A}_1] + 2 [\bar{A}_2] (k-k_\ell) + 3 [\bar{A}_3] (k-k_\ell)^2 \quad (4.38)$$

For option H3, where  $[AIC(k)]$  and  $[W(k)]$  are input as the product  $[AW(k)] = [AIC(k)] [W(k)]$  and option H5, where  $[H]^T$ ,  $[AIC(k)]$  and  $[W(k)]$  are input as the product  $[HAW(k)] = [H]^T [AIC(k)] [W(k)]$ , the  $\bar{A}$  quantities are obtained from corresponding  $\bar{AW}$  and  $\bar{HAW}$  quantities which in turn are obtained from all  $\bar{AW}(k_\ell)$  and  $\bar{HAW}(k_\ell)$  according to cubic spline theory.

For options H1b, H2b and H4b, where  $[DX]$  and  $[DZ]$  are input, rather than  $[W(k)]$ , the formulas equivalent to equations 4.37 and 4.38 are more complicated as shown in Section 4.3.2.

The matrices  $[A(k_\ell)]$  and  $[A'(k_\ell)]$  in equations 4.34 and 4.35, the matrices  $[\bar{A}_0]$ ,  $[\bar{A}_1]$ ,  $[\bar{A}_2]$  and  $[\bar{A}_3]$  in equations 4.37 and 4.38 and the  $\bar{A}$  matrices used in Options H1b, H2b and H4b are formed in the flutter optimization module by appropriate pre- and postmultiplications, depending on the option. Always, however there is an aerodynamics input recognizable that is related to the set of discrete  $k$  values  $k_\ell$  and that is generated outside the flutter optimization module.

**4.3.1 Basis for Comparison.** The count formulas in Section 4.3.2 are developed according to the following rules. Accuracy is not used as a basis for comparison. For given dimensions of the matrices used to form the generalized aerodynamic force coefficients matrix the accuracy of the aerodynamics depends on the theory used, including the method for integrating or lumping the pressures and interpolating and differentiating the structural displacements.

1. Only the computational requirements for repetitive calculations inside the flutter optimization module are determined. Preparing the input for the flutter optimization module is non-repetitive and

its impact on the total optimization is small. Note, for example, that interpolation by cubic spline requires considerably more computations outside the flutter optimization module than does interpolation by cubic polynomial.

2. The number of computational operations associated with matrix-matrix multiplications and scalar-matrix multiplications are counted. In general one computational operation is one multiplication followed by one addition. Since an addition takes only about 1/10 the time of a multiplication, in the few places where only multiplication occurs, the multiplication is counted as one operation. Additions are not counted when occurring alone. Generating one element of a real matrix product requires  $N$  operations if  $N$  is the number of columns in the premultiplier. Generating one complex element of the product of a real and a complex matrix takes  $2N$  operations. One complex element of the product of two complex matrices takes  $4N$  operations.
3. Matrices  $[H]$  and  $[W(k)]$ , both with  $N$  columns, may be sparsely populated. This enters into the computational requirements by indicating that each row in these matrices has  $\phi N$  non-zero elements. Multiplication by zero is assumed to cost no computer time.
4. The number of computational operations (Oper) includes only the formation of the matrices that enter into the interpolation formulas, but not the operations for the actual interpolation.
5. The number of matrix element words (one word for real, two words for complex element) to be stored (Stor) for easy access by the flutter optimization module is determined.
6. Since it may be desirable to keep the matrices involved in the interpolation of the aerodynamics in core during the optimization procedure, the required core (Cor) space in terms of the number of matrix element words is determined.
7. During an optimization the value of  $k$  may drift from one interval to the next. If the matrices involved in the interpolation are kept in core, then it is of interest to know how much of the core must be replaced as  $k$  drifts to the next interval. This is called the read-in count (Read).

4.3.2 Count Formulas. Consistent with the preceding general discussion, specific count formulas for several options are developed in this section. The following may assist in interpreting the formulas.



The matrices  $[A(k_\ell)]$  and  $[A'(k_\ell)]$  are the matrix of generalized aerodynamic force coefficients and its derivative to be evaluated outside the flutter optimization program for discrete values  $k = k_\ell$ . These matrices are to be used in a Lagrangian interpolation formula corresponding to a cubic polynomial. The number of  $k_\ell$  values required is  $n_\lambda = n_\delta + 3$ , where  $n_\delta$  is the number of  $k$  intervals needed. The matrices  $[A(k)]$  and  $[A'(k)]$  are the interpolated aerodynamics matrix and its derivative. These symbols are used in connection with the cubic spline options.

The barred quantities:  $[\overline{AIC}_j]$ ,  $[\overline{AW}_j]$ ,  $[\overline{HA}_j]$  and  $[\overline{HAW}_j]$ , to be used in equations of the form given in equation 4.37, are functions of all  $k_\ell$  values and a set of four ( $j=0,1,2,3$ ) is computed for each interval, as required, with the help of cubic spline formulas.

Additional shortened notation is used:

$$[HA(k)] = [H]^T [AIC(k)] , [AW(k)] = [AIC(k)][W(k)] \quad (4.39)$$

Multiplications contributing to the number of computational operations to be performed in the flutter optimization program are indicated by a heavy dot. The sequence of multiplication is indicated by Arabic numerals. Products, already evaluated, but re-used in another part of the same option lack the multiplication dot and are assembled within matrix brackets; e.g.

$$[z]^T [H]^T .$$

The dimensions of the matrices to be multiplied are summarized below. The symbol  $\phi N$  is used when the actual dimension is  $N$ , but there is a possibility that the matrix is sparse.

The interpretation of the matrix dimensions used is as follows:

M: number of modes used in flutter analysis

N: number of discrete degrees of freedoms: structural displacements

K: number of lumped aerodynamic forces

D: number of downwash collocation points

$\phi$ : fraction of non zero elements in rows of  $[W(k)]$  and  $[H]$ .

Options H1 and H2

$$\begin{matrix} [\bar{z}]^T & [H]^T & [AIC(k)] & [W(k)] & [\bar{z}] \\ (M,N) & (\phi N,K) & (K,D) & (D,\phi N) & (N,M) \end{matrix} \quad (4.40)$$

Option H3

$$\begin{matrix} [\bar{z}]^T & [H]^T & [AW(k)] & [\bar{z}] \\ (M,N) & (\phi N,K) & (K,N) & (N,M) \end{matrix} \quad (4.41)$$

Option H4

$$\begin{matrix} [\bar{z}]^T & [HA(k)] & [W(k)] & [\bar{z}] \\ (M,N) & (N,D) & (D,\phi N) & (N,M) \end{matrix} \quad (4.42)$$

Option H5

$$\begin{matrix} [\bar{z}]^T & [HAW(k)] & [\bar{z}] \\ (M,N) & (N,N) & (N,M) \end{matrix} \quad (4.43)$$

Option H1a Cubic Polynomial

$$\begin{matrix} [A(k_\ell)] = [\bar{z}]^T \cdot [H]^T \cdot [AIC(k_\ell)] \cdot [W(k_\ell)] \cdot [\bar{z}] \\ \underbrace{\hspace{10em}}_1 \hspace{1em} \underbrace{\hspace{10em}}_2 \\ \underbrace{\hspace{10em}}_3 \hspace{1em} \underbrace{\hspace{10em}}_4 \end{matrix}$$

$$\begin{matrix} [A'(k_\ell)] = [\bar{z}]^T [H]^T \cdot [AIC'(k_\ell)] \cdot [W(k_\ell)] [\bar{z}] + \\ \underbrace{\hspace{10em}}_5 \hspace{1em} \underbrace{\hspace{10em}}_6 \end{matrix}$$

$$\begin{matrix} + [\bar{z}]^T [H]^T [AIC(k_\ell)] \cdot i [DZ] [\bar{z}] \\ \underbrace{\hspace{10em}}_7 \hspace{1em} \underbrace{\hspace{10em}}_8 \end{matrix}$$

Computational operations:

sequence number	1 $\phi NMK$	2 $2n_{\lambda} \phi NMD$	3 $2n_{\lambda} KMD$	4 $4n_{\lambda} DM^2$
sequence number	5 $2n_{\lambda} KMD$	6 $4n_{\lambda} DM^2$	7 $\phi NMD$	8 $2n_{\lambda} DM^2$

Total number of operations:

$$\text{Oper Hla Poly} = \phi NMK + (2n_{\lambda} + 1) \phi NMD + 4n_{\lambda} KMD + 10n_{\lambda} DM^2$$

Required input quantities:  $[H]$ ,  $[AIC(k_{\ell})]$ ,  $[A'(k_{\ell})]$ ,  $[W(k_{\ell})]$  and  $[DZ]$

$$\text{Stor Hla Poly} = \phi NK + 4n_{\lambda} KD + (2n_{\lambda} + 1) \phi ND$$

Required for interpolation:  $[A(k_{\ell})]$  and  $[A'(k_{\ell})]$  for four  $k$  values

$$\text{Cor Hla Poly} = 16 M^2$$

Required read-in for new interval:  $[A(k_{\ell})]$  and  $[A'(k_{\ell})]$  for one  $k$  value

$$\text{Read Hla Poly} = 4 M^2$$

#### Option Hla Cubic Spline

$$[A(k)] = \sum_{j=0}^3 \underbrace{[\bar{z}]^T \cdot [H]^T \cdot [\overline{AIC}_j]}_1 \cdot \underbrace{[W(k)] \cdot [\bar{z}]}_2 (k - k_{\ell})^j$$

$\underbrace{\hspace{10em}}_3$   
 $\underbrace{\hspace{15em}}_4$

$$\begin{aligned}
[A'(k)] = & \sum_{j=0}^3 \left[ \left[ \bar{z} \right]^T [H]^T [\overline{AIC}_j] [W(k)] \left[ \bar{z} \right] \right] j (k-k_\ell)^{j-1} + \\
& + \underbrace{\sum_{j=0}^3 \left[ \left[ \bar{z} \right]^T [H]^T [\overline{AIC}_j] \right]}_6 \cdot \underbrace{\left[ DX \right] \cdot \left[ \bar{z} \right]}_5 i (k-k_\ell)^j
\end{aligned}$$

In the H1a sequence  $[W(k)]$  is computed for each interpolated  $k$  value, which is accomplished by evaluating  $[W(k)] = [DX] + ik [DZ]$ . Therefore, Option H1a Cubic Spline is identical to Option H1b Cubic Spline.

Option H1b Cubic Polynomial

$$\begin{aligned}
[A(k_\ell)] = & \underbrace{\left[ \bar{z} \right]^T \cdot [H]^T \cdot [AIC(k_\ell)]}_3 \cdot \underbrace{[DX] \cdot \left[ \bar{z} \right]}_2 + \\
& + k_\ell \underbrace{\left[ \left[ \bar{z} \right]^T [H]^T [AIC(k_\ell)] i [DZ] \left[ \bar{z} \right] \right]}_{10}
\end{aligned}$$

$$\begin{aligned}
 [A'(k_\ell)] = & \underbrace{\left[ \underbrace{[\bar{Z}]^T [H]^T}_{5} \cdot [AIC'(k_\ell)] \right]}_6 \cdot \underbrace{[DX] [\bar{Z}]}_7 + \\
 & + k_\ell \underbrace{\left[ \underbrace{[\bar{Z}]^T [H]^T [AIC'(k_\ell)]}_{11} \right]}_{12} \cdot \underbrace{[i[DZ][\bar{Z}]]}_{8} + \underbrace{\left[ \underbrace{[\bar{Z}]^T [H]^T [AIC(k_\ell)]}_{9} \cdot i \cdot [DZ] \cdot [\bar{Z}] \right]}_{8}
 \end{aligned}$$

Computational operations:

sequence	1	2	3	4	5	6
number	$\phi MNK$	$\phi MND$	$2n_\lambda KMD$	$2n_\lambda DM^2$	$2n_\lambda KMD$	$2n_\lambda DM^2$
sequence	7	8	9	10	11	12
number	$\phi NMD$	0	$2n_\lambda DM^2$	$2n_\lambda M^2$	$2n_\lambda DM^2$	$2n_\lambda M^2$

Total number of operations:

$$\text{Oper Hlb Poly} = \phi MNK + 2\phi NMD + 4n_\lambda KMD + 8n_\lambda DM^2 + 4n_\lambda M^2$$

Required input quantities:  $[H]$ ,  $[AIC(k_\ell)]$ ,  $[AIC'(k_\ell)]$ ,  $[DX]$  and  $[DZ]$

$$\text{Stor Hlb Poly} = \phi NK + 4n_\lambda KD + 2\phi ND$$

Required for interpolation:  $[A(k_\ell)]$  and  $[A'(k_\ell)]$  for four  $k$  values

$$\text{Cor Hlb Poly} = 16 M^2$$

Required read-in for new interval:  $[A(k_\ell)]$  and  $[A'(k_\ell)]$  for one  $k$  value

$$\text{Read Hlb Poly} = 4M^2$$

Option H1b Cubic Spline

$$[A(k)] = \sum_{j=0}^3 \underbrace{\underbrace{[\bar{z}]^T \cdot [H]^T}_{1} \cdot [\overline{AIC}_j]}_3 \cdot \underbrace{[DX] \cdot [\bar{z}]}_2 (k-k_\ell)^j +$$

$$+ \sum_{j=0}^3 \underbrace{\underbrace{[\bar{z}]^T [H]^T [\overline{AIC}_j]}_5 \cdot [DZ] \cdot [\bar{z}]}_6 i k (k-k_\ell)^j$$

$$[A'(k)] = \sum_{j=0}^3 \left[ [\bar{z}]^T [H]^T [\overline{AIC}_j] [DX] [\bar{z}] \right] j (k-k_\ell)^{j-1} +$$

$$+ \sum_{j=0}^3 \left[ [\bar{z}]^T [H]^T [\overline{AIC}_j] [DZ] [\bar{z}] \right] i (k-k_\ell)^j +$$

$$+ \sum_{j=0}^3 \left[ [\bar{z}]^T [H]^T [\overline{AIC}_j] [DZ] [\bar{z}] \right] i k j (k-k_\ell)^{j-1}$$

Computational operations:

sequence	1	2	3	4	5	6
number	$\phi N M K$	$\phi N D M$	$8 n_\delta K D M$	$8 n_\delta D M^2$	$\phi N D M$	$8 n_\delta D M^2$

Total number of operations:

$$\text{Oper H1b Spline} = \phi N M K + 2 \phi N D M + 8 n_\delta K D M + 16 n_\delta D M^2$$

Required input quantities (for definition of barred quantities see the beginning of this section and equation 4.37):

$$[H] , [\overline{AIC}_0] , [\overline{AIC}_1] , [\overline{AIC}_2] , [\overline{AIC}_3] , [DX] \text{ and } [DZ]$$

$$\text{Stor Hlb Spline} = \phi NK + 8n_{\delta}KD + 2_{\delta}ND$$

Required for interpolation:

$$[\bar{z}]^T [H]^T [\overline{AIC}_j] [DX] [\bar{z}] \text{ and } [\bar{z}]^T [H]^T [\overline{AIC}_j] [DZ] [\bar{z}] ,$$

four of each per interval ( $j=0,1,2,3$ ).

$$\text{Cor Hlb Spline} = 16 M^2$$

Required read-in for new interval: Replace complete core

$$\text{Read Hlb Spline} = 16 M^2$$

#### Option H2a Cubic Polynomial

$$[A(k_{\ell})] = \underbrace{[\bar{z}]^T \cdot [H]^T}_{1} \cdot \underbrace{[\overline{AIC}(k_{\ell})]}_{3} \cdot \underbrace{[W(k_{\ell})] \cdot [\bar{z}]}_{2}$$

4

$$[A'(k_{\ell})] = \underbrace{[\bar{z}]^T [H]^T}_{6} \cdot \underbrace{[\overline{AIC}'(k_{\ell})] \cdot [W(k_{\ell})] [\bar{z}]}_{5} + \underbrace{[\bar{z}]^T [H]^T}_{8} \cdot \underbrace{[\overline{AIC}(k_{\ell})] \cdot i[DZ] [\bar{z}]}_{7}$$

9

Computational operations:

sequence	1	2	3	4	
number	$\phi N M K$	$2 n_{\lambda} \phi N M D$	$4 n_{\lambda} K D M$	$2 n_{\lambda} K M^2$	
sequence	5	6	7	8	9
number	$4 n_{\lambda} K D M$	$2 n_{\lambda} K M^2$	$\phi N M D$	$2 n_{\lambda} K D M$	$2 n_{\lambda} K M^2$

Total number of operations:

$$\text{Oper H2a Poly} = \phi N M K + (2 n_{\lambda} + 1) \phi N M D + 10 n_{\lambda} K D M + 6 n_{\lambda} K M^2$$

Required input quantities:  $[H]$ ,  $[AIC(k_{\ell})]$ ,  $[AIC'(k_{\ell})]$ ,  $[W(k_{\ell})]$  and  $[DZ]$

$$\text{Stor H2a Poly} = \phi N K + 4 n_{\lambda} K D + (2 n_{\lambda} + 1) \phi N D$$

Required for interpolation:  $[A(k_{\ell})]$  and  $[A'(k_{\ell})]$  for four  $k$  values

$$\text{Cor H2a Poly} = 16 M^2$$

Required read-in for new interval:  $[A(k_{\ell})]$  and  $[A'(k_{\ell})]$  for one  $k$  value

$$\text{Read H2a Poly} = 4 M^2$$

Option H2a Cubic Spline

$$[A(k)] = \sum_{j=0}^3 \underbrace{[\bar{z}]^T \cdot [H]^T}_{1} \cdot \underbrace{[AIC_j] \cdot [W(k)] \cdot [\bar{z}]}_{2} \underbrace{(k-k_{\ell})^j}_{3}$$

4



$$\begin{aligned}
[A'(k)] = & \sum_{j=0}^3 \left[ [\bar{z}]^T [H]^T [\overline{AIC}_j] [W(k)] [\bar{z}] \right] j (k-k_\ell)^{j-1} + \\
& + \sum_{j=0}^3 \underbrace{\left[ [\bar{z}]^T [H]^T \right] \cdot [\overline{AIC}_j] \cdot \underbrace{[DZ] \cdot [\bar{z}]}_5 \cdot i (k-k_\ell)^j}_{\underbrace{\hspace{10em}}_6} \\
& \underbrace{\hspace{10em}}_7
\end{aligned}$$

In the H2a sequence  $[W(k)]$  is computed for each interpolated  $k$  value which is accomplished by evaluating  $[W(k)] = [DX] + ik [DZ]$ . Therefore Option H2a Cubic Spline is identical to Option H2b Cubic Spline.

Option H2b Cubic Polynomial

$$\begin{aligned}
[A(k_\ell)] = & \underbrace{[\bar{z}]^T \cdot [H]^T}_1 \cdot [\overline{AIC}(k_\ell)] \cdot \underbrace{[DX] \cdot [\bar{z}]}_2 + \\
& \underbrace{\hspace{10em}}_3 \\
& \underbrace{\hspace{10em}}_4 \\
& + k_\ell \cdot \underbrace{\left[ [\bar{z}]^T [H]^T [\overline{AIC}(k_\ell)] \cdot i [DZ] [\bar{z}] \right]}_{11}
\end{aligned}$$

$$\begin{aligned}
[A'(k_\ell)] = & \underbrace{\left[ [\bar{z}]^T [H]^T \right] \cdot [AIC'(k_\ell)] \cdot [DX] [\bar{z}]}_6 + \\
& + k_\ell \cdot \underbrace{\left[ [\bar{z}]^T [H]^T \right] \cdot [AIC'(k_\ell)] \cdot [i[DZ] [\bar{z}]}_{12} + \\
& \underbrace{\phantom{+ k_\ell \cdot} }_{13} \\
& \underbrace{\phantom{+ k_\ell \cdot} }_{14} \\
& + \underbrace{\left[ [\bar{z}]^T [H]^T \right] \cdot [AIC(k_\ell)] \cdot i \cdot [DZ] \cdot [\bar{z}]}_{10}
\end{aligned}$$

Computational operations:

sequence number	1 $\phi\text{NMK}$	2 $\phi\text{NMD}$	3 $2n_\lambda \text{KDM}$	4 $2n_\lambda \text{KM}^2$	5 $2n_\lambda \text{KDM}$
sequence number	6 $2n_\lambda \text{KM}^2$	7 $\phi\text{NMD}$	8 0	9 $2n_\lambda \text{KDM}$	10 $2n_\lambda \text{KM}^2$
sequence number	11 $2n_\lambda \text{M}^2$	12 $2n_\lambda \text{KDM}$	13 $2n_\lambda \text{KM}^2$	14 $2n_\lambda \text{M}^2$	

Total number of operations

$$\text{Oper H2b Poly} = \phi\text{NMK} + 2\phi\text{NMD} + 8n_\lambda \text{KDM} + 8n_\lambda \text{KM}^2 + 4n_\lambda \text{M}^2$$

Required input quantities:  $[H]$ ,  $[AIC(k_\ell)]$ ,  $[AIC(k_\ell)]$ ,  $[DX]$  and  $[DZ]$

$$\text{Stor H2b Poly} = \phi NK + 4n_\lambda KD + 2\phi ND$$

Required for interpolation:  $[A(k_\ell)]$  and  $[A'(k_\ell)]$  for four  $k$  values

$$\text{Cor H2b Poly} = 16 M^2$$

Required read-in for new interval:  $[A(k_\ell)]$  and  $[A'(k_\ell)]$  for one  $k$  value

$$\text{Read H2b Poly} = 4 M^2$$

### Option H2b Cubic Spline

$$\begin{aligned}
 [A(k)] = & \sum_{j=0}^3 \underbrace{[\bar{z}]^T \cdot [H]^T}_{1} \cdot \underbrace{[\overline{AIC}_j] \cdot [DX] \cdot [\bar{z}]}_{2} (k-k_\ell)^j + \\
 & \underbrace{\hspace{10em}}_{3} \\
 & \underbrace{\hspace{10em}}_{4} \\
 & + \sum_{j=0}^3 \underbrace{\left[ [\bar{z}]^T [H]^T \right]}_{5} \cdot \underbrace{[\overline{AIC}_j]}_{6} \cdot \underbrace{[DZ] \cdot [\bar{z}]}_{7} ik (k-k_\ell)^j
 \end{aligned}$$

$$\begin{aligned}
[A'(k)] = & \sum_{j=0}^3 \left[ [\bar{z}]^T [H]^T [\overline{AIC}_j] [DX] [\bar{z}] \right] j (k-k_\ell)^{j-1} + \\
& + \sum_{j=0}^3 \left[ [\bar{z}]^T [H]^T [\overline{AIC}_j] [DZ] [\bar{z}] \right] i (k-k_\ell)^j + \\
& + \sum_{j=0}^3 \left[ [\bar{z}]^T [H]^T [\overline{AIC}_j] [DZ] [\bar{z}] \right] i k j (k-k_\ell)^{j-1}
\end{aligned}$$

Computational operations:

sequence	1	2	3	4	5	6	7
number	$\phi\text{NMK}$	$\phi\text{NDM}$	$8n_\delta\text{KDM}$	$8n_\delta\text{KM}^2$	$\phi\text{NMD}$	$8n_\delta\text{KDM}$	$8n_\delta\text{KM}^2$

Total number of operations

$$\text{Oper H2b Spline} = \phi\text{NMK} + 2\phi\text{NMD} + 16n_\delta\text{KDM} + 16n_\delta\text{KM}^2$$

Required input quantities:  $[H]$ ,  $[\overline{AIC}_0]$ ,  $[\overline{AIC}_1]$ ,  $[\overline{AIC}_2]$ ,  $[\overline{AIC}_3]$ ,  $[DX]$  and  $[DZ]$

$$\text{Stor H2b Spline} = \phi\text{NK} + 8n_\delta\text{KD} + 2\phi\text{ND}$$

Required for interpolation:  $[\bar{z}]^T [H]^T [\overline{AIC}_j] [DX] [\bar{z}]$  and

$[\bar{z}]^T [H]^T [\overline{AIC}_j] [DZ] [\bar{z}]$ , four of each per interval.

$$\text{Core H2b Spline} = 16 M^2$$

Required read-in for new interval: Replace complete core

$$\text{Read H2b Spline} = 16 M^2$$

### Option H3 Cubic Polynomial

$$[A(k_\ell)] = \underbrace{[\bar{z}]^T \cdot [H]^T}_1 \cdot \underbrace{[AW(k_\ell)]}_2 \cdot [\bar{z}]$$

3

$$[A'(k_\ell)] = \underbrace{\left[ [\bar{z}]^T \quad [H]^T \right]}_5 \cdot \underbrace{[AW'(k_\ell)]}_4 \cdot [\bar{z}]$$

Computational operations:

sequence	1	2	3	4	5
number	$\phi N M K$	$2n_\lambda K N M$	$2n_\lambda K M^2$	$2n_\lambda K N M$	$2n_\lambda K M^2$

Total number of operations:

$$\text{Oper H3 Poly} = \phi N M K + 4n_\lambda K N M + 4n_\lambda K M^2$$

Required input quantities:  $[H]$ ,  $[AW(k_\ell)]$  and  $[AW'(k_\ell)]$

$$\text{Stor H3 Poly} = \phi N K + 4n_\lambda K N$$

Required for interpolation:  $[A(k_\ell)]$  and  $[A'(k_\ell)]$  for four  $k$  values

$$\text{Cor H3 Poly} = 16 M^2$$

Required read-in for new interval:  $[A(k_\ell)]$  and  $[A'(k_\ell)]$  for one  $k$  value

$$\text{Read H3 Poly} = 4 M^2$$

### Option H3 Cubic Spline

$$[A(k)] = \sum_{j=0}^3 \underbrace{[\bar{z}]^T \cdot [H]^T \cdot [\bar{A}W_j]}_3 \cdot [\bar{z}] (k-k_\ell)^j$$

$$[A'(k)] = \sum_{j=0}^3 \left[ [\bar{z}]^T [H]^T [\bar{A}W_j] [\bar{z}] \right] j (k-k_\ell)^{j-1}$$

Computational operations:

sequence	1	2	3
number	$\phi NMK$	$8n_\delta KNM$	$8n_\delta KM^2$

Total number of operations:

$$\text{Oper H3 Spline} = \phi NMK + 8n_\delta KNM + 8n_\delta KM^2$$

Required input quantities:  $[H]$ ,  $[\bar{A}W_0]$ ,  $[\bar{A}W_1]$ ,  $[\bar{A}W_2]$  and  $[\bar{A}W_3]$

$$\text{Stor H3 Spline} = \phi NM + 8n_\delta KN$$

Required for interpolation: four  $[\bar{z}]^T [H]^T [\bar{A}W_j] [\bar{z}]$

$$\text{Cor H3 Spline} = 8 M^2$$

Required read-in for new interval: Replace complete core

$$\text{Read H3 Spline} = 8 M^2$$

### Option H4a Cubic Polynomial

$$[A(k_\ell)] = \underbrace{[\bar{z}]^T \cdot [HA(k_\ell)]}_{1} \cdot \underbrace{[W(k_\ell)]}_{2} \cdot [\bar{z}]$$

3

$$[A'(k_\ell)] = \underbrace{[\bar{z}]^T \cdot [HA'(k_\ell)]}_{4} \cdot \underbrace{[W(k_\ell)] [\bar{z}]}_{5} + \underbrace{\left[ [\bar{z}]^T [HA(k)] \right]}_{7} \cdot i \cdot \underbrace{[DZ] [\bar{z}]}_{6}$$

8

Computational operations:

sequence number	1 $2n_\lambda \text{MND}$	2 $2n_\lambda \phi \text{NMD}$	3 $4n_\lambda \text{DM}^2$	4 $2n_\lambda \text{MND}$
sequence number	5 $4n_\lambda \text{DM}^2$	6 $\phi \text{NDM}$	7 0	8 $2n_\lambda \text{DM}^2$

Total number of operations:

$$\text{Oper H4a Poly} = 4n_\lambda \text{MND} + (2n_\lambda + 1) \phi \text{NMD} + 10n_\lambda \text{DM}^2$$

Required input quantities:  $[HA(k_\ell)]$ ,  $[HA'(k_\ell)]$ ,  $[W(k_\ell)]$  and  $[DZ]$

$$\text{Stor H4a Poly} = 4n_\lambda \text{ND} + (2n_\lambda + 1) \phi \text{ND}$$

Required for interpolation:  $[A(k_\ell)]$  and  $[A'(k_\ell)]$  for four  $k$  values

$$\text{Cor H4a Poly} = 16 \text{M}^2$$

Required read-in for new interval:  $[A(k_\ell)]$  and  $[A'(k_\ell)]$  for one  $k$  value

$$\text{Read H4a Poly} = 4 \text{M}^2$$

Option H4a Cubic Spline

$$[A(k)] = \sum_{j=0}^3 \underbrace{[\bar{z}]^T \cdot [\overline{HA}_j]}_1 \cdot \underbrace{[W(k)] \cdot [\bar{z}]}_2 (k-k_\ell)^j$$

3

$$[A'(k)] = \sum_{j=0}^3 \left[ [\bar{z}]^T [\overline{HA}_j] [W(k)] [\bar{z}] \right] j (k-k_\ell)^{j-1} +$$

$$+ \sum_{j=0}^3 \underbrace{\left[ [\bar{z}]^T [\overline{HA}_j] \right] \cdot \underbrace{[DZ] \cdot [\bar{z}]}_4}_5 i (k-k_\ell)^j$$

In the H4a sequence  $[W(k)]$  is computed for each interpolated  $k$  value, which is accomplished by evaluating  $[W(k)] = [DX] + ik [DZ]$ . Therefore Option H4a Cubic Spline is identical to Option H4b Cubic Spline.

Option H4b Cubic Polynomial

$$[A(k_\ell)] = \underbrace{[\bar{z}]^T \cdot [HA(k_\ell)]}_1 \cdot \underbrace{[DX] [\bar{z}]}_2 + k_\ell \cdot \underbrace{\left[ [\bar{z}]^T [HA(k_\ell)] i [DZ] [\bar{z}] \right]}_7$$

3



$$\begin{aligned}
[A'(k_\ell)] = & \underbrace{[\bar{z}]^T \cdot [HA'(k_\ell)] \cdot [DX] [\bar{z}]}_{11} + k_\ell \cdot \underbrace{[\bar{z}]^T [HA'(k_\ell)] \cdot [i[DZ][\bar{z}]]}_{9} + \\
& + \underbrace{[\bar{z}]^T [HA(k_\ell)] \cdot i \cdot [DZ] \cdot [\bar{z}]}_{6}
\end{aligned}$$

Computational operations:

sequence	1	2	3	4	5	6	7
number	$2n_\lambda \text{MND}$	$\phi \text{NMD}$	$2n_\lambda \text{DM}^2$	$\phi \text{NMD}$	0	$2n_\lambda \text{DM}^2$	$2n_\lambda \text{M}^2$
sequence	8	9	10	11			
number	$2n_\lambda \text{DM}^2$	$2n_\lambda \text{M}^2$	$2n_\lambda \text{MND}$	$2n_\lambda \text{DM}^2$			

Total number of operations:

$$\text{Oper H4b Poly} = 2\phi \text{NMD} + 4n_\lambda \text{NMD} + 8n_\lambda \text{DM}^2 + 4n_\lambda \text{M}^2$$

Required input quantities:  $[HA(k_\ell)]$ ,  $[HA'(k_\ell)]$ ,  $[DX]$  and  $[DZ]$

$$\text{Stor H4b Poly} = 4n_\lambda \text{ND} + 2\phi \text{ND}$$

Required for interpolation:  $[A(k_\ell)]$  and  $[A'(k_\ell)]$  for four  $k$  values

$$\text{Cor H4b Poly} = 16 \text{M}^2$$

Required read-in for new interval:  $[A(k_\ell)]$  and  $[A'(k_\ell)]$  for one  $k$  value

$$\text{Read H4b Poly} = 4 \text{M}^2$$

Option H4b Cubic Spline

$$\begin{aligned}
 [A(k)] = & \sum_{j=0}^3 \underbrace{[\bar{z}]^T \cdot [\overline{HA}_j]}_1 \cdot \underbrace{[DX] \cdot [\bar{z}]}_2 (k-k_\ell)^j + \\
 & + \sum_{j=0}^3 \underbrace{\left[ [\bar{z}]^T [\overline{HA}_j] \right] \cdot \underbrace{[DZ] \cdot [\bar{z}]}_4}_{5} i k (k-k_\ell)^j \\
 [A'(k)] = & \sum_{j=0}^3 \left[ [\bar{z}]^T [\overline{HA}_j] [DX] [\bar{z}] \right]_j (k-k_\ell)^{j-1} + \\
 & + \sum_{j=0}^3 \left[ [\bar{z}]^T [\overline{HA}_j] [DZ] [\bar{z}] \right]_i (k-k_\ell)^j + \\
 & + \sum_{j=0}^3 \left[ [\bar{z}]^T [\overline{HA}_j] [DZ] [\bar{z}] \right]_{ik} j (k-k_\ell)^{j-1}
 \end{aligned}$$

Computational operations

sequence	1	2	3	4	5
number	$8n_\delta \text{NMD}$	$\phi \text{NMD}$	$8n_\delta \text{DM}^2$	$\phi \text{NMD}$	$8n_\delta \text{DM}^2$

Total number of operations:

$$\text{Oper H4b Spline} = 2\phi \text{NMD} + 8n_\delta \text{NMD} + 16n_\delta \text{DM}^2$$

Required input quantities:  $[\overline{HA}_0]$ ,  $[\overline{HA}_1]$ ,  $[\overline{HA}_2]$ ,  $[\overline{HA}_3]$ ,  $[DX]$  and  $[DZ]$

$$\text{Stor H4b Spline} = 8n_\delta ND + 2\phi ND$$

Required for interpolation:  $[\overline{z}]^T [\overline{HA}_j] [DX] [\overline{z}]$  and  $[\overline{z}]^T [\overline{HA}_j] [DZ] [\overline{z}]$

$$\text{Cor H4b Spline} = 16 M^2$$

Required read-in for new interval: Replace complete core

$$\text{Read H4b Spline} = 16 M^2$$

#### Option H5 Cubic Polynomial

$$[A(k_\ell)] = [\overline{z}]^T \cdot \underbrace{[HAW(k_\ell)]}_1 \cdot [\overline{z}]$$

2

$$[A'(k_\ell)] = [\overline{z}]^T \cdot \underbrace{[HAW'(k_\ell)]}_3 \cdot [\overline{z}]$$

4

Computational operations:

sequence	1	2	3	4
number	$2n_\lambda N^2 M$	$2n_\lambda NM^2$	$2n_\lambda N^2 M$	$2n_\lambda NM^2$

Total number of operations:

$$\text{Oper H5 Poly} = 4n_\lambda N^2 M + 4n_\lambda NM^2$$

Required input quantities:  $[HAW(k_\ell)]$  and  $[HAW'(k_\ell)]$

$$\text{Stor H5 Poly} = 4n_\lambda N^2$$

Required for interpolation:  $[A(k_\ell)]$  and  $[A'(k_\ell)]$  for four  $k$  values

$$\text{Cor H5 Poly} = 16 M^2$$

Required read in for new interval:  $[A(k_\ell)]$  for one  $k$  value

$$\text{Read H5 Poly} = 4 M^2$$

Option H5 Cubic Spline

$$[A(k)] = \sum_{j=0}^3 \underbrace{[\bar{z}]^T \cdot \underbrace{[\overline{HAW}_j]}_1 \cdot [\bar{z}]}_2 (k-k_\ell)^j$$

$$[A'(k)] = \sum_{j=0}^3 \left[ [\bar{z}]^T [\overline{HAW}_j] [\bar{z}] \right] j (k-k_\ell)^{j-1}$$

Computational operations

sequence	1	2
number	$8n_\delta N^2 M$	$8n_\delta N M^2$

Total number of operations:

$$\text{Oper H5 Spline} = 8n_\delta N^2 M + 8n_\delta N M^2$$

Required input quantities:  $[\overline{HAW}_0]$ ,  $[\overline{HAW}_1]$ ,  $[\overline{HAW}_2]$  and  $[\overline{HAW}_3]$

$$\text{Stor H5 Spline} = 8n_\delta N^2$$

Required for interpolation:  $[\bar{z}]^T [\overline{HAW}_j] [\bar{z}]$

$$\text{Cor H5 Spline} = 8 M^2$$

Required read-in for new interval: Replace complete core

$$\text{Read H5 Spline} = 8 M^2$$

4.3.3 Comparisons. The formulas developed in the preceding section are tabulated in Tables 4-1, 4-2 and 4-3. A summary of the comparisons is presented in Reference 1 and is repeated below.

Input Storage Requirements - If the number of  $k$  intervals to be used is three or more, cubic polynomial interpolation for arbitrary values of  $k$  requires less input storage than cubic spline interpolation for all options H1 through H5.

Core Space - For options H5 and H3 cubic polynomial interpolation for arbitrary  $k$  requires two times as much core space as cubic spline interpolation. For all other options both methods of interpolation require the same core space.

Read-In - Cubic polynomial interpolation for arbitrary  $k$  requires less read-in than cubic spline interpolation as the value of  $k$  moves into an adjacent interval.

Number of Computational Operations - Cubic polynomial interpolation for arbitrary value of  $k$  requires fewer computational operations than cubic spline interpolation under the following conditions:

options H3 and H5: if the number of  $k$  intervals is more than three.

options H1 and H4: if the number of  $k$  intervals is more than four.

option H2: if the number of  $k$  intervals is more than five.

Which of the options H1 through H5 is most efficient is strongly affected by the dimensions of the matrices. From equation 4.40 it can be seen that if  $K$  and  $D$  are small compared with  $N$  it becomes advantageous to perform the multiplications  $[\bar{z}]^T [H]^T$  and  $[W(k)] [\bar{z}]$  in the flutter optimization module. If  $K$  and  $D$  are equal to  $N$  or larger, then it becomes

advantageous to form the product  $[H]^T [AIC(k)] [W(k)]$  outside the flutter module. The relationships defining when one option is better than another are complicated and no simple criteria have become apparent.

# Number of Computational Operations

Option	Number of Operations
H1a Polynomial	$\phi NMK + (2n_\delta + 7) \phi NMD + 4(n_\delta + 3)KMD + 10(n_\delta + 3)DM^2$
H1b Polynomial	$\phi NMK + 2\phi NMD + 4(n_\delta + 3)KMD + 8(n_\delta + 3)DM^2 + 4(n_\delta + 3)M^2$
H1a Spline } H1b Spline }	$\phi NMK + 2\phi NMD + 8n_\delta KMD + 16n_\delta DM^2$
H2a Polynomial	$\phi NMK + (2n_\delta + 7) \phi NMD + 10(n_\delta + 3)KMD + 6(n_\delta + 3)KM^2$
H2b Polynomial	$\phi NMK + 2\phi NMD + 8(n_\delta + 3)KMD + 8(n_\delta + 3)KM^2 + 4(n_\delta + 3)M^2$
H2a Spline } H2b Spline }	$\phi NMK + 2\phi NMD + 16n_\delta KMD + 16n_\delta KM^2$
H3 Polynomial	$\phi NMK + 4(n_\delta + 3)KNM + 4(n_\delta + 3)KM^2$
H3 Spline	$\phi NMK + 8n_\delta KNM + 8n_\delta KM^2$
H4a Polynomial	$4(n_\delta + 3)NMD + (2n_\delta + 7) \phi NMD + 10(n_\delta + 3)DM^2$
H4b Polynomial	$2\phi NMD + 4(n_\delta + 3)NMD + 8(n_\delta + 3)DM^2 + 4(n_\delta + 3)M^2$
H4a Spline } H4b Spline }	$2\phi NMD + 8n_\delta NMD + 16n_\delta DM^2$
H5 Polynomial	$4(n_\delta + 3)N^2M + 4(n_\delta + 3)NM^2$
H5 Spline	$8n_\delta N^2M + 8n_\delta NM^2$

- M number of modes used in flutter analysis  
 N number of discrete degrees of freedom: structural displacements  
 K number lumped aerodynamic forces  
 D number of downwash collocation points  
 $\phi$  fraction of non-zero elements in rows of  $[W(k)]$  and  $[H]$   
 $n_\delta$  number of  $k$  intervals to be considered

Table 4-1: Comparison of Number of Computational Operations in Forming the Aerodynamics Matrices

# Storage Required for Input Quantities

Option	Storage
H1a Polynomial	$\phi NK + 4(n_\delta + 3)KD + (4n_\delta + 13) \phi ND$
H1b Polynomial	$\phi NK + 4(n_\delta + 3)KD + 2\phi ND$
H1a Spline } H1b Spline }	$\phi NK + 8n_\delta KD + 2\phi ND$
H2a Polynomial	$\phi NK + 4(n_\delta + 3)KD + (2n_\delta + 7) \phi ND$
H2b Polynomial	$\phi NK + 4(n_\delta + 3)KD + 2\phi ND$
H2a Spline } H2b Spline }	$\phi NK + 8n_\delta KD + 2\phi ND$
H3 Polynomial	$\phi NK + 4(n_\delta + 3)KN$
H3 Spline	$\phi NK + 8n_\delta KN$
H4a Polynomial	$4(n_\delta + 3)ND + (2n_\delta + 7) \phi ND$
H4b Polynomial	$4(n_\delta + 3)ND + 2\phi ND$
H4a Spline } H4b Spline }	$8n_\delta ND + 2\phi ND$
H5 Polynomial	$4(n_\delta + 3)N^2$
H5 Spline	$8n_\delta N^2$

N number of discrete degrees of freedom: structural displacements  
 K number of lumped aerodynamic forces  
 D number of downwash collocation points  
 $\phi$  fraction of non-zero elements in rows of  $[W(k)]$  and  $[H]$   
 $n_\delta$  number of  $k$  intervals to be considered

Table 4-2: Aerodynamics Input Storage Requirements

# Core Space for Interpolation and Read-in for New k Interval

Core	Core Space	Read-in
H1a Polynomial	$16M^2$	$4M^2$
H1b Polynomial	$16M^2$	$4M^2$
H1a Spline } H1b Spline }	$16M^2$	$16M^2$
H2a Polynomial	$16M^2$	$4M^2$
H2b Polynomial	$16M^2$	$4M^2$
H2a Spline } H2b Spline }	$16M^2$	$16M^2$
H3 Polynomial	$16M^2$	$4M^2$
H3 Spline	$8M^2$	$8M^2$
H4a Polynomial	$16M^2$	$4M^2$
H4b Polynomial	$16M^2$	$4M^2$
H4a Spline } H4b Spline }	$16M^2$	$16M^2$
H5 Polynomial	$16M^2$	$4M^2$
H5 Spline	$8M^2$	$8M^2$

M number of modes used in flutter analysis

Table 4-3: Core Space and Read-In Required for Interpolation of the Aerodynamics Matrices

## 5. THEORETICAL CONSIDERATIONS RELATED TO OPTIMIZATION WITH FLUTTER CONSTRAINTS

### 5.1 Introduction

For a better understanding and appreciation of the methods of optimization with flutter constraints discussed in Reference 1 it is helpful to reduce the problem of optimization to simple terms.



The elementary considerations presented in this section are based on the assumption of satisfying a minimum flutter speed requirement for several flight conditions and minimizing the structural weight.

The design variables are structural sizings, such as cross-sectional areas of caps, skin and web thicknesses, as well as non-structural mass (ballast). These design variables lead to a linear relation between the total mass and the design variables. It is convenient to define each design variable by the mass it represents. Then:

$$M = \sum_{i=1}^n m_i \quad (5.1)$$

where  $M$  is the total mass associated with the design variables  $m_i$ .

Structural elements with a stiffness defined by two design variables lead to a nonlinear relation between  $M$  and  $m_i$ ; for instance, if a beam cap is defined by a width and a height as design variables. Another example of a nonlinear relation between total mass and a design variable is the stiffness of a constant width bending element; its mass is proportional to the cubic root of the design variable. Adequate finite element structural representation is possible without such nonlinearities and thus equation 5.1 has sufficient generality.

If all structural coordinates defining a structural model are retained in the vibration and flutter analysis, the stiffness matrix is a linear function of the design variables. If for reasons of practicability a number of structural coordinates must be eliminated this relationship may become nonlinear. Any nonlinear relationship between the stiffness matrix and the design variables is implicit in the relationship between the flutter speed and the design variables,  $V(m_i)$ , which is nonlinear even if the stiffness is a linear function of the design variables.

If there is only one flutter speed constraint the optimization problem can be stated as: minimize  $M(m_i)$  while the condition  $V(m_i) = V_R$  is satisfied, where  $V_R$  is the required flutter speed. With the help of the Lagrangian multiplier  $\lambda$  this problem can be formulated as  $n+1$  equations with the  $n+1$  unknowns  $m_i$  and  $\lambda$ :

$$\left. \begin{aligned} \frac{\partial}{\partial m_i} \left[ M(m_i) + \lambda \{ V(m_i) - V_R \} \right] &= 0 \\ V(m_i) - V_R &= 0 \end{aligned} \right\} \quad (5.2)$$

Equations 5.2 reduce to

$$\frac{\partial V}{\partial m_i} = \frac{-1}{\lambda} \quad \text{and} \quad V = V_R \quad (5.3)$$

The interpretation of the equations 5.3 is that at the optimum condition all partial derivatives of the flutter speed with respect to the design variables are equal. The equation is only valid for free design variables; i.e. variables which, at the optimum point, have not reached a minimum or maximum allowable value.

In a realistic design environment equations 5.3 cannot be solved directly. The usual procedure is that starting with a non-optimum design a sequence of design changes is generated which leads to the optimum design.

In the following sections such sequences of design changes are discussed. Separate sections are devoted to increasing the flutter speed to a desired flutter speed with a minimum mass penalty, to minimizing the mass while keeping the flutter speed constant, and to considering multiple flutter speed constraints.

## 5.2 Increasing the Flutter Speed with Minimum Mass Penalty

Let  $V_U$  be the unsatisfactory flutter speed of an initial design and  $V_R$  the required higher flutter speed. The problem of increasing the flutter speed from  $V_U$  to  $V_R$  can be illustrated with a system with two design variables  $m_1$  and  $m_2$ . Figure 5-1 is obtained from a realistic numerical example. The structure is a simple beam representation of the wing of a subsonic transport. Design variable  $m_1$  defines the incremental torsional stiffness over the center one third of the span of the exposed wing and  $m_2$  the incremental torsional stiffness of the outer one third. Contours for constant flutter speed and constant total mass increment over a reference mass are indicated. By definition  $m_1 = 0$  and  $m_2 = 0$  for  $V_U = 450$  KEAS. The desired flutter speed is  $V_R = 550$  KEAS. Sequences of design changes in the direction of the  $V_R = 550$  KEAS curve are indicated. Four different starting points are examined: O, A, B and C, and it is assumed that the starting point defines minimum allowable values for  $m_1$  and  $m_2$ . Two optimum design points are marked in Figure 5-1. One is a free optimum, not affected by a minimum allowable value of either  $m_1$  or  $m_2$ . It is the tangential point of the 550 KEAS contour and a constant total mass line. The other optimum design point on the 550 KEAS contour is not an extremum in the mathematical sense; it is a minimum mass design consistent with the constraint imposed by the minimum value of  $m_1$ .

Consider two criteria for determining the direction in the  $m_1, m_2$  design plane: direction of steepest ascent and direction of maximum  $\frac{dV}{dM}$ .

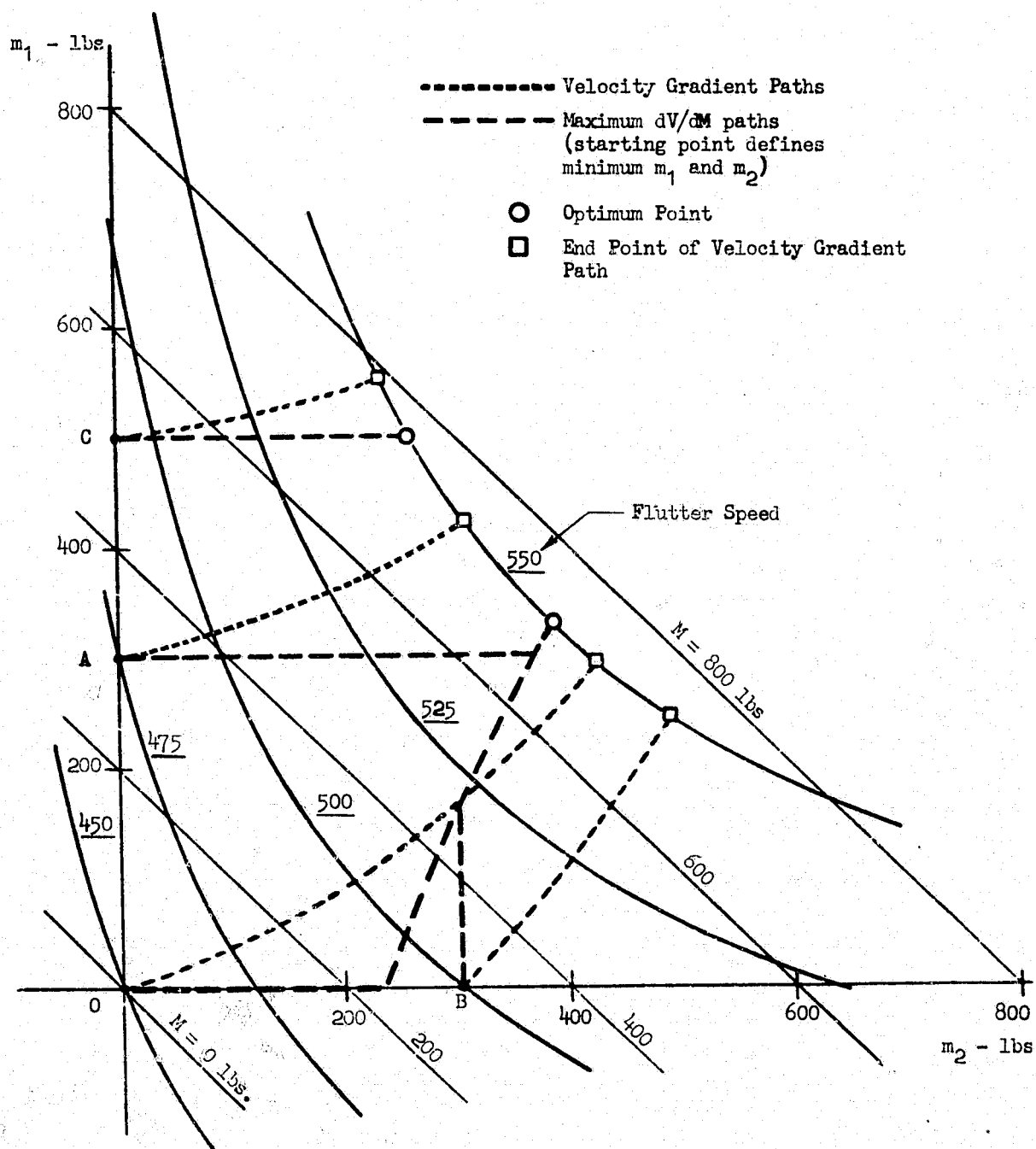


Figure 5-1: Resizing Paths as Functions of Two Design Variables.

5.2.1 Direction of steepest ascent. The direction of steepest ascent is a direction, defined by the direction cosines  $\frac{dm_1}{ds}$  and  $\frac{dm_2}{ds}$  ( $ds^2 = dm_1^2 + dm_2^2$ ) for which  $\frac{dV}{ds}$  is maximum. It can be shown that this maximum occurs when

$$\frac{dm_1}{ds} = \frac{\partial V / \partial m_1}{\sqrt{\left(\frac{\partial V}{\partial m_1}\right)^2 + \left(\frac{\partial V}{\partial m_2}\right)^2}}, \quad \frac{dm_2}{ds} = \frac{\partial V / \partial m_2}{\sqrt{\left(\frac{\partial V}{\partial m_1}\right)^2 + \left(\frac{\partial V}{\partial m_2}\right)^2}} \quad (5.4)$$

and

$$\frac{dV}{ds} = \sqrt{\left(\frac{\partial V}{\partial m_1}\right)^2 + \left(\frac{\partial V}{\partial m_2}\right)^2} \quad (5.5)$$

Equations 5.4 define a ratio between two infinitesimal increments of  $m_1$  and  $m_2$  given by:

$$\frac{dm_1}{dm_2} = \frac{\partial V / \partial m_1}{\partial V / \partial m_2} \quad (5.6)$$

The tangent to a constant flutter speed contour is defined by.

$$\frac{dm_1}{dm_2} = -\frac{\partial V / \partial m_2}{\partial V / \partial m_1} \quad (5.7)$$

Thus the direction of steepest ascent, at any point, is perpendicular to the constant flutter speed contours. A finite design increment step in that direction can be defined by:

$$\begin{Bmatrix} \Delta m_1 \\ \Delta m_2 \end{Bmatrix} = \Delta s \begin{Bmatrix} dm_1/ds \\ dm_2/ds \end{Bmatrix} = \frac{\Delta s}{\sqrt{\left(\frac{\partial V}{\partial m_1}\right)^2 + \left(\frac{\partial V}{\partial m_2}\right)^2}} \begin{Bmatrix} \partial V / \partial m_1 \\ \partial V / \partial m_2 \end{Bmatrix} \quad (5.8)$$

The column  $\begin{Bmatrix} \partial V / \partial m_1 \\ \partial V / \partial m_2 \end{Bmatrix}$  is called the gradient of  $V$  with respect to the design variables.

In equation 5.8 the stepsize  $\begin{Bmatrix} \Delta m_1 \\ \Delta m_2 \end{Bmatrix}$  is defined by the magnitude of  $\Delta s$ , the distance travelled in the  $m_1, m_2$  plane. With the help of equation 5.5 equation 5.8 can be written as:

$$\begin{Bmatrix} \Delta m_1 \\ \Delta m_2 \end{Bmatrix} = \frac{\Delta V}{\sqrt{\left(\frac{\partial V}{\partial m_1}\right)^2 + \left(\frac{\partial V}{\partial m_2}\right)^2}} \begin{Bmatrix} \partial V / \partial m_1 \\ \partial V / \partial m_2 \end{Bmatrix} \quad (5.9)$$

which relates the step size to an increment in flutter speed.

Design paths following the direction of steepest ascent are indicated in Figure 5-1. It is noted that they do not lead to the optimum design point.

5.2.2 Direction of maximum  $\frac{dV}{dM}$ . The direction of maximum  $\frac{dV}{dM}$  is the

direction in which the increment of the flutter speed per unit increment of total mass is maximum.

At an arbitrary point in the  $m_1, m_2$  design plane there is no direction  $\frac{dm_1}{dm_2}$  for which  $\frac{dV}{dM}$  is an extremum. Consider, for instance, point A as an initial design point and move along an  $M=\text{constant}$  contour towards the

$m_2$  axis:  $V$  increases,  $M$  remains the same, thus  $\frac{dV}{dM} = \infty$ . A design change

along an  $M=\text{constant}$  contour, however, is not feasible in view of the original assumption that at the initial design point  $m_1$  and  $m_2$  are at their minimum values. From simple geometric considerations it follows that

the maximum feasible value of  $\frac{dV}{dM}$  is in the direction of the coordinate for

which  $\frac{\partial V}{\partial m}$  is largest. At point A  $\frac{\partial V}{\partial m_2} > \frac{\partial V}{\partial m_1}$  and a design change parallel to the  $m_2$  axis results in the largest gain in flutter speed per unit mass

added until the locus for which  $\frac{\partial V}{\partial m_2} = \frac{\partial V}{\partial m_1}$  is reached. Since this locus

connects points representing minimum total mass designs as a function of flutter speed, following this locus leads to the optimum design at  $V_R = 550$  KEAS.

Along this locus the direction  $\frac{dm_1}{dm_2}$  is determined from the requirement

$$d\left(\frac{\partial V}{\partial m_1}\right) = d\left(\frac{\partial V}{\partial m_2}\right), \text{ from which } \frac{\partial^2 V}{\partial m_1^2} dm_1 + \frac{\partial^2 V}{\partial m_1 \partial m_2} dm_2 = \frac{\partial^2 V}{\partial m_2 \partial m_1} dm_1 + \frac{\partial^2 V}{\partial m_2^2} dm_2$$

and

$$\frac{dm_1}{dm_2} = \frac{\partial^2 V / \partial m_2^2 - \partial^2 V / \partial m_1 \partial m_2}{\partial^2 V / \partial m_1^2 - \partial^2 V / \partial m_2 \partial m_1} \quad (5.10)$$

Paths following maximum feasible  $\frac{dV}{dM}$  from the initial design points

O and B are also indicated in Figure 5-1.

Note that an initial design defined by point C, leads to a minimum weight design at which  $V = 550$  that is not an extremum condition, due to the minimum value imposed on  $m_1$ .

**5.2.3 Discussion.** Figure 5-1 shows that following a direction defined by the gradient of the flutter speed, in the case shown, leads to a design configuration close to optimum. This result cannot be generalized. In a multi-design variable system it is not unlikely that several design variables remain at their initial value for the optimum design. If that is the case the result of following the steepest ascent direction generally will be farther from the optimum design than is indicated in figure 5-1.

Generalizing equation 5.9 to  $n$  design variables leads to the following expression for the column of design variable increments:

$$\{\Delta m_i\} = \frac{\Delta V}{\sqrt{\sum_1^n \left(\frac{\partial V}{\partial m_i}\right)^2}} \left\{ \frac{\partial V}{\partial m_i} \right\} \quad (5.11)$$

Generalization of the path of maximum  $\frac{dV}{dM}$  approach to many design variables takes the following, albeit impracticable form.

Assume, as before, that at the initial design point all design variables are at their minimum allowable value. Obtain the largest gain in flutter speed per unit mass by increasing the design variable corresponding to the

largest  $\frac{\partial V}{\partial m}$ . As this design variable is increased it most likely becomes

less effective in raising the flutter speed and after a certain amount of change there may be two design variables with the largest value of

$\frac{\partial V}{\partial m}$ . Then these two variables can be changed according to the relationship

given by equation 5.10. When three  $\frac{\partial V}{\partial m}$ 's are equal the three associated

variables can be changed in a ratio that follows from a reasoning similar to

the reasoning that led to equation 5.10. Successively more  $\frac{\partial V}{\partial m}$ 's

become equal. When the desired flutter speed is reached the  $\frac{\partial V}{\partial m}$ 's associated

with the design variables that have been increased above their minimum value are all equal, as was found before as a condition of optimality (equations 5.3).

It should be remarked that the procedure is only presented to provide some insight in the process that leads to an optimum design.

Although it is noted that a path of steepest ascent does not lead to the optimum design, it does lead to a favorable initial configuration from which to start minimizing the total mass while keeping the flutter speed at  $V_R$ . It was found (Reference 1, Appendix A) that an efficient approach to defining such an initial configuration is to use a resizing column

$$\{\Delta m_i\} = C \left\{ \frac{\partial V}{\partial m_i} \right\}, \quad (5.12)$$

where  $C$  is determined by requiring that the flutter speed is raised to  $V_R$  in one step. It leads to an initial weight only little higher than that which results from a multi-step approach, but at a considerable reduction in computational effort.

The presentation in the preceding sections is considerably simplified by defining the design variables in terms of mass units. In Reference 1

it is shown that the distribution in a resizing column defined by  $\frac{\partial V}{\partial P_i}$ ,

where  $P_i$  is a generalized design variable:  $P_i = \frac{m_i}{C_i}$ , depends on the

individual scaling of the design variables. E.g.  $C_i$  for skin thickness

as a design variable is different from a  $C_i$  for cap area as a design variable. Since obviously the best distribution in a resizing column should be independent of the scaling, the effect of scaling must be eliminated by a normalization. The simplest normalization is to let  $C_i = 1$ .

### 5.3 Minimizing the Total Mass at Constant Flutter Speed

5.3.1 Elementary Steps. Minimizing the structural mass while keeping the flutter speed constant has considerable practical significance. In Section 5.2 it was shown that the rather straightforward method of steepest ascent does not lead to the optimum design. However, the method of following a path of maximum  $\frac{dV}{dM}$  is rather intricate.

There are practical methods of minimizing the structural mass which may be used once the flutter speed has the desired value. Several are discussed in detail in Reference 1. It is instructive to consider an elementary approach, based on simple reasoning, since it provides insight into practical methods.

The elementary approach is based on the following basic design change: take away material where it causes the smallest decrease of the flutter speed per unit mass removed and add material where it causes the largest increase of the flutter speed per unit mass added.

The mathematical formulation is as follows:

Let the flutter speed,  $V$ , be given by:

$$V = V(m_1, m_2, \dots, m_n) = V_R \quad (5.13)$$

Then

$$\frac{\partial V}{\partial m_1} dm_1 + \frac{\partial V}{\partial m_2} dm_2 + \dots = 0 \quad (5.14)$$

Let  $\partial V / \partial m_1 > 0$  and  $\partial V / \partial m_2 > 0$ ,  $\partial V / \partial m_1$  the largest and  $\partial V / \partial m_2$  the smallest of all  $\partial V / \partial m$ 's. Reducing  $m_2$  to  $m_2 - \Delta m_2$  causes the smallest decrease in flutter speed per unit mass removed. From equation 5.14 it



follows that the amount to be added to  $m_1$  to maintain approximately constant flutter speed is:

$$\Delta m_1 = \frac{\partial V / \partial m_2}{\partial V / \partial m_1} \Delta m_2 \quad (5.15)$$

The total mass change is

$$\Delta M = \Delta m_1 - \Delta m_2 = \left( \frac{\partial V / \partial m_2}{\partial V / \partial m_1} - 1 \right) \Delta m_2 \quad (5.16)$$

which is a reduction.

Small steps  $\Delta m_1$ ,  $\Delta m_2$  can be taken in succession and all derivatives recalculated after each step. As long as  $\partial V / \partial m_1$  and  $\partial V / \partial m_2$  remain the largest and the smallest of the  $\partial V / \partial m$ 's the same design variables are changed. As soon as a third derivative  $\partial V / \partial m_3$  equals either  $\partial V / \partial m_1$  or  $\partial V / \partial m_2$  it must be considered in determining the next step in a manner similar to what is discussed in Section 5.2.2. If  $m_2$  reaches its minimum allowable value the design variable with the next to the smallest value of  $\partial V / \partial m$  is used instead. It is not difficult to see that a stage is reached in which the  $\partial V / \partial m$ 's associated with design variables that are not at their minimum allowable value are all equal. Nothing can be gained by increasing one and decreasing another variable by a small amount: the optimum design is attained.

Of interest is the case in which some  $\frac{\partial V}{\partial m}$ 's are negative. With

$\partial V / \partial m_2$  being the smallest  $\partial V / \partial m$  this implies  $\partial V / \partial m_2 < 0$  and decreasing  $m_2$  increases the flutter speed. The initial resizing steps, therefore, should be to reduce  $m_2$  and let the flutter speed increase. Successive steps of reducing  $m_2$  can be taken until occurrence of either one of the following:  $m_2$  reaches its minimum allowable value,  $\partial V / \partial m_2$  equals another negative  $\partial V / \partial m$ , or  $\partial V / \partial m_2$  has become positive and as a result the gain in flutter speed is cancelled.

**5.3.2 Composite Resizing Columns.** It is recognized that the elementary steps described in the preceding section as well as those described in Section 5.2.2 are an inefficient approach to optimization. It is also recognized that a resizing column proportional to the gradient of the flutter speed (equation 5.12) to a certain extent satisfies the basic notion of adding most material where it causes the largest increase of the flutter speed per unit mass. However, material is also added where, according to the reasoning in the preceding section it should be subtracted. This is indeed the very reason why following the direction of steepest ascent does not lead to an

optimum. What is needed is a resizing column that removes the most mass where it causes the smallest decrease of the flutter speed per unit mass. A logical candidate is a resizing column that is the sum of two linearly independent distributions and leads to the desired effect. Equation 5.17 shows a simple approach to following this concept:

$$\{\Delta m_i\} = A \left\{ \frac{\partial V}{\partial m_i} \right\} - B \left\{ \left( \frac{\partial V}{\partial m} \right)_{\max} - \frac{\partial V}{\partial m_i} \right\} \quad (5.17)$$

This can be written as:

$$\{\Delta m_i\} = (A + B) \left\{ \frac{\partial V}{\partial m_i} \right\} - B \left( \frac{\partial V}{\partial m} \right)_{\max} \{1\} \quad (5.18)$$

or

$$\{\Delta m_i\} = A' \left\{ \frac{\partial V}{\partial m_i} \right\} - B' \{1\} \quad (5.19)$$

which is a form identical to the resizing column for the gradient projection search at constant flutter speed of Rudisill and Bhatia (Reference 6 and Reference 1, Section 6.2).

Mass removal where it causes the smallest decrease of the flutter speed per unit mass is emphasized if the second column of equation 5.17 is

replaced by  $\left\{ \frac{1}{\partial V / \partial m_i} \right\}$ :

$$\{\Delta m_i\} = A \left\{ \frac{\partial V}{\partial m_i} \right\} - B \left\{ \frac{1}{\partial V / \partial m_i} \right\} \quad (5.20)$$

The constant A is related to the total mass addition,  $W_1$ , associated with the first column:

$$A = \frac{W_1}{[1] \left\{ \partial V / \partial m_i \right\}} \quad (5.21)$$

The constant B can be determined from the condition

$$\Delta V = \left[ \frac{\partial V}{\partial m_i} \right] \{\Delta m_i\} = 0 \quad (5.22)$$

which would result in zero change in flutter speed if V were a linear function of the design variables  $m_i$ .

Applying equation 5.22 to equation 5.20, for example, leads to:

$$B = \frac{A}{n} \sum_{i=1}^n \left( \frac{\partial V}{\partial m_i} \right)^2 \quad (5.23)$$

Due to the presence of nonlinearities, however, equation 5.23 does, in general, lead to a finite change in flutter speed.

A condition somewhat more complicated than equation 5.22 follows from the requirement that the flutter speed is to be kept exactly constant. It is determined by solving equation 5.24 for  $k$  and  $B$  according to the method of Incremented Flutter Analysis

$$D \left( g, \gamma, k, V, \{m_i\}, \frac{W_1}{[1]} \frac{\{\partial V / \partial m_i\}}{\{\partial V / \partial m_i\}}, B \{VM\} \right) = 0 \quad (5.24)$$

where  $\{VM\}$  corresponds to the second column of either equation 5.17 or equation 5.20.

The value  $W_1$  can be chosen arbitrarily or it can be based on a step size criterion, such as discussed in Reference 1, Section 6.6.

The resizing column defined by equation 5.20 has been used successfully for a case in which all  $\frac{\partial V}{\partial m}$ 's had positive values. Results are presented in Reference 1.

When there are negative  $\partial V / \partial m$ 's the  $\partial V / \partial m_i$  contribution in equations 5.17 and 5.20 tends to remove most mass where it increases the flutter speed most. On the other hand the  $\frac{1}{\partial V / \partial m_i}$  column in equation 5.20 adds to design variables with negative  $\frac{\partial V}{\partial m}$  and tends to outweigh the  $\frac{\partial V}{\partial m_i}$  contribution.

Figure 5-2 shows, graphically, possibilities of modifying the columns in equation 5.20 to accommodate negative  $\partial V / \partial m$ 's. The elements are arranged in the order of increasing value of  $\partial V / \partial m$  on the vertical axis. The value of each element is measured along the horizontal axis.

Figure 5-2(a) shows, for reference, the components of the resizing column of equation 5-20, but for positive  $\partial V / \partial m$ 's only. Figure 5-2(b) extends into the range of negative  $\frac{\partial V}{\partial m}$ 's. For negative values of  $\partial V / \partial m$

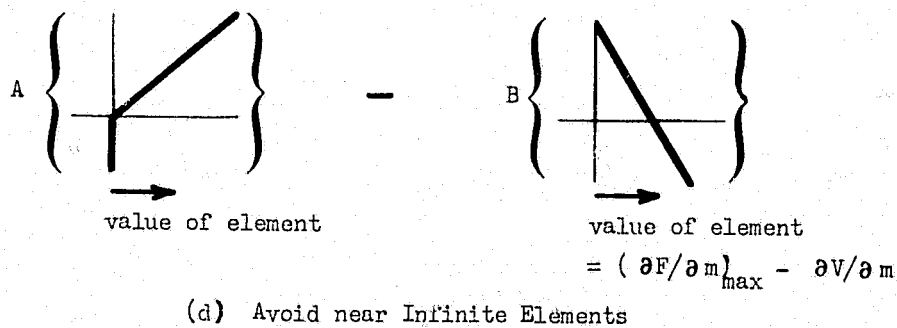
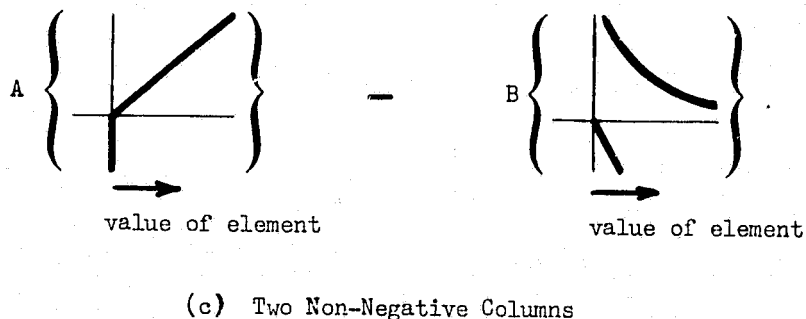
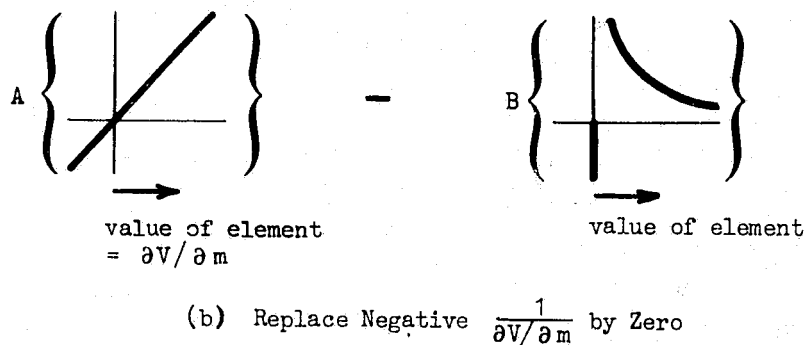
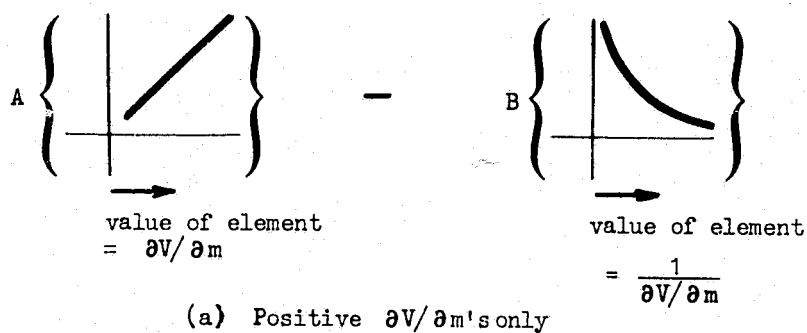


Figure 5-2: Composite Resizing Columns.

the elements in the second column are set to zero to avoid the problem associated with large negative values of  $\frac{1}{\partial V / \partial m}$ .

Figure 5-2(c) is a variation of Figure 5-2(b) in that the negative  $\frac{\partial V}{\partial m}$ 's have been moved, with opposite sign, from the first to the second column. Thus neither column has negative elements. This has advantages for including minimum size constraints in resizing algorithms.

Equation 5-17 is represented in Figure 5-2(d), also with the modification that avoids negative elements in the component columns.

Since in an actual airplane design several or many  $\partial V / \partial m$ 's may be very close to zero, the column  $\left\{ \frac{1}{\partial V / \partial m_i} \right\}$  may contain extremely large elements. If this would constitute a problem a limit could be set on the value of  $\frac{1}{\partial V / \partial m}$ , or the problem could be avoided by using the resizing column of Figure 5-2(d).

## 5.4 Multiple Flutter Speed Constraints

**5.4.1 Introduction.** When the flutter analysis of an airplane design indicates significantly unsatisfactory flutter characteristics, it often happens that for different loading conditions (full fuel, no fuel, full pay load, etc.) and/or Mach number several instances of flutter speeds less than the required  $V_R$  occur. Even for one loading condition and one Mach number several zero modal dampings at speeds below  $V_R$  may occur. When resizing the structure to satisfy the flutter speed constraints one may think of trying to move all zero damping intersections up to  $V_R$ . Whether or not this is possible is irrelevant from a practical point of view, since it does not necessarily lead to the lightest structure, as is shown in the next paragraph.

Assume a structure with two flutter speeds,  $V_1$  and  $V_2$ , both below  $V_R$ . In some fashion (uniform increase of stiffness level, steepest ascent, or other) the structure is modified such that  $V_1 = V_R$ . Assume that now  $V_2 > V_R$ . A one-flutter-speed-constraint mass minimization, keeping  $V_1 = V_R$ , can now be performed. Consider the case for which  $V_2$  remains above  $V_R$  until the minimum mass for  $V_1 = V_R$  is reached. Then that is the minimum mass for which the flutter constraints  $V_1 \geq V_R$  and  $V_2 \geq V_R$  are satisfied.

To lower  $V_2$  to  $V_R$  while keeping  $V_1 = V_R$ , if it is at all possible, requires an increase in mass since the minimum weight for which  $V_1 = V_R$  is already attained.

If in minimizing the mass for  $V_1 = V_R$ ,  $V_2$  decreases to  $V_R$ , then both flutter speeds should be maintained at  $V_R$  until a minimum mass condition is reached.

When there are more than two flutter speeds to contend with, minimizing the mass is usually accomplished by the following procedure.

Considering the complete set of flutter characteristics (complete  $f$ - $g$ - $V$  diagrams for various Mach numbers and loading conditions) of the initial structure, it is estimated which of the zero damping intersections is most critical. The structure is modified to move this intersection to  $V_R$ . If there still remains an intersection at a speed below  $V_R$  the structure is further modified. The result is an initial non-optimum structure with  $V_1 = V_R$  and  $V_i \geq V_R$  ( $i=2,3,\dots,v$ ). A one-flutter-speed-constraint mass minimization, keeping  $V_1 = V_R$ , is performed until a minimum mass design is reached, or until another flutter speed, say  $V_2$ , decreases to  $V_2 = V_R$ . In the latter case the weight minimization is continued with two flutter speed constraints:  $V_1 = V_2 = V_R$ . Eventually a third and more speeds may be drawn into the minimization procedure.

The following section provides mathematical background for the above discussion.

**5.4.2 Mathematical background.** If two flutter speed constraints are active the optimization problem can be stated as: minimize  $M(m_i)$  while the conditions  $V_1(m_i) = V_{R1}$  and  $V_2(m_i) = V_{R2}$  are satisfied. (Note that different flight conditions may have different  $V_R$ ). With the help of the Lagrangian multipliers  $\lambda_1$  and  $\lambda_2$  this problem can be formulated as  $n+2$  equations with the  $n+2$  unknowns  $m_i$ ,  $\lambda_1$  and  $\lambda_2$ :

$$\left. \begin{aligned} \frac{\partial}{\partial m_i} \left[ M(m_i) + \lambda_1 \{V_1(m_i) - V_{R1}\} + \lambda_2 \{V_2(m_i) - V_{R2}\} \right] &= 0 \\ V_1(m_i) - V_{R1} &= 0 \qquad V_2(m_i) - V_{R2} = 0 \end{aligned} \right\} \quad (5.25)$$

These equations reduce to

$$\lambda_1 \frac{\partial V_1}{\partial m_i} + \lambda_2 \frac{\partial V_2}{\partial m_i} + 1 = 0 \qquad i=1,2,\dots,n \quad (5.26)$$

and

$$V_1 = V_{R1} \quad , \quad V_2 = V_{R2} \quad (5.27)$$

Consider three of the equations 5.26, say for  $i = 1, 2$  and  $3$ :

$$\left. \begin{aligned} \lambda_1 \frac{\partial V_1}{\partial m_1} + \lambda_2 \frac{\partial V_2}{\partial m_1} + 1 &= 0 \\ \lambda_1 \frac{\partial V_1}{\partial m_2} + \lambda_2 \frac{\partial V_2}{\partial m_2} + 1 &= 0 \\ \lambda_1 \frac{\partial V_1}{\partial m_3} + \lambda_2 \frac{\partial V_2}{\partial m_3} + 1 &= 0 \end{aligned} \right\} \quad (5.28)$$

Only non-zero values of  $\lambda_1$  and  $\lambda_2$  can satisfy equation 5.26 for finite values of  $\frac{\partial V_1}{\partial m_i}$  and  $\frac{\partial V_2}{\partial m_i}$ . The condition for non-zero solutions for  $\lambda_1$  and  $\lambda_2$  follows from equations 5.28:

$$\begin{vmatrix} 1 & 1 & 1 \\ \frac{\partial V_1}{\partial m_1} & \frac{\partial V_1}{\partial m_2} & \frac{\partial V_1}{\partial m_3} \\ \frac{\partial V_2}{\partial m_1} & \frac{\partial V_2}{\partial m_2} & \frac{\partial V_2}{\partial m_3} \end{vmatrix} = 0 \quad (5.29)$$

Since the subscripts 1, 2 and 3 are arbitrarily assigned, equation 5.29 must be satisfied for all possible combinations of three design variables.

In the case of one flutter speed constraint equation 5.29 reduces to:

$$\begin{vmatrix} 1 & 1 \\ \frac{\partial V_1}{\partial m_1} & \frac{\partial V_1}{\partial m_2} \end{vmatrix} = 0 \quad (5.30)$$

from which:

$$\frac{\partial V_1}{\partial m_1} = \frac{\partial V_1}{\partial m_2}$$

which corresponds to the first equation of equations 5.3.

Extension of equation 5.29 to  $v$  flutter speed constraints leads to:

$$\begin{vmatrix} 1 & 1 & . & . & . & 1 \\ \frac{\partial V_1}{\partial m_1} & \frac{\partial V_1}{\partial m_2} & . & . & . & \frac{\partial V_1}{\partial m_{v+1}} \\ \frac{\partial V_2}{\partial m_1} & \frac{\partial V_2}{\partial m_2} & . & . & . & \frac{\partial V_2}{\partial m_{v+1}} \\ . & . & . & . & . & . \\ \frac{\partial V_v}{\partial m_1} & \frac{\partial V_v}{\partial m_2} & . & . & . & \frac{\partial V_v}{\partial m_{v+1}} \end{vmatrix} = 0 \quad (5.31)$$

which must be satisfied for any set  $m_1, m_2, \dots, m_{v+1}$  within  $m_1, m_2, \dots, m_n$  ( $n \geq v+1$ ).

Equation 5.31 is satisfied if any row of the determinant has equal elements.

$$\frac{\partial V_1}{\partial m_1} = \frac{\partial V_1}{\partial m_2} = \dots = \frac{\partial V_i}{\partial m_n} \quad (5.32)$$

This is the minimum mass condition for one flutter speed constraint. It means that if the mass is minimized while satisfying one flutter speed constraint  $V_1 = V_{R1}$  and all other flutter speeds exceed their associated  $V_R$ , the minimum mass for  $V_i \geq V_{Ri}$  ( $i=1,2,\dots,v$ ) is obtained regardless of the values of the derivatives in the other rows. This conclusion was reached by informal reasoning in Section 5.4.1.



Another special condition for which equation 5.31 is satisfied is:

$$k_1 \frac{\partial V_1}{\partial m_1} + k_2 \frac{\partial V_2}{\partial m_1} = k_1 \frac{\partial V_1}{\partial m_2} + k_2 \frac{\partial V_2}{\partial m_2} = \dots = k_1 \frac{\partial V_1}{\partial m_n} + k_2 \frac{\partial V_2}{\partial m_n} \quad (5.33)$$

where  $k_1$  and  $k_2$  are arbitrary constants.

This condition occurs when the mass is minimized while keeping two flutter speeds at their corresponding  $V_R$ . If all other flutter speeds equal or exceed their associated  $V_R$ , then the minimum mass for  $V_i \geq V_{Ri}$

( $i=1,2,\dots,v$ ) is obtained regardless of the values of the derivatives in the rows not represented in equation 5.33. The reasoning can be expanded to cover any number of active flutter speed constraints.

The same restriction is imposed on equation 5.31 that is imposed on equations 5.3: equation 5.31 is only valid for free design variables; i.e. design variables which, at the optimum point have not reached a minimum or maximum allowable value.

**5.4.3 Discussion.** Considerable background on optimization with one flutter speed constraint has been published and more is provided in Reference 1. The elementary considerations related to the one-flutter-speed-constraint optimization in preceding sections, therefore, relate directly to published material.

Although several methods of optimization appear sufficiently generally formulated to be applicable to the case of multiple flutter speed constraints the literature does not provide any examples. The following discussion, therefore, is rather speculative.

#### 5.4.3.1 Increasing the Flutter Speeds

In the case of one flutter speed constraint a simple resizing column is defined, proportional to the initial gradient of the flutter speed (equation 5.12), to increase the flutter speed to  $V_R$  in one step. Numerical examples have shown that this provides a favorable initial design from which to start a mass minimization process at constant flutter speed.

The resizing distribution defined by equation 5.12 follows the notion of adding to each variable an amount proportional to its effectiveness in increasing the flutter speed. Extending this notion to the case of several unsatisfied flutter speed constraints a resizing column could be conceived in which each element is proportional to the effectiveness of the associated design variable in increasing all submarginal flutter speeds. This could

lead to a resizing column that is proportional to the sum of the gradients of the flutter speeds:

$$\{\Delta m_i\} = C \left\{ \sum_{j=1}^u \frac{\partial V_j}{\partial m_i} \right\} \quad (5.34)$$

( $V_j$  = flutter speeds below  $V_R$ ;  $j = 1, 2, \dots, u$ )

However, one flutter speed may be close to its  $V_R$ , another far below. Obviously, design variables favoring the latter should be weighted more heavily than variables favoring the former. This leads to a resizing column based on a weighted sum of the gradients of the flutter speed:

$$\{\Delta m_i\} = C \left\{ \sum_{j=1}^u (V_{Rj} - V_j) \frac{\partial V_j}{\partial m_i} \right\} \quad (5.35)$$

The scalar  $C$  should be determined such that one flutter speed  $V_1 = V_{R1}$  and all other flutter speeds  $V_j \geq V_{Rj}$  ( $j=2, 3, \dots, u$ ).

It is emphasized that the authors are not aware of any numerical evaluation of the suggested procedure.

#### 5.4.3.2 Minimizing the Mass

Consider first the case of one active flutter speed constraint:

$$V_1 = V_{R1}, \quad V_j \geq V_{Rj} \quad (j=2, 3, \dots, v).$$

Use the resizing column of equation 5.20, with  $A$  replaced by the value given by equation 5.21, and compute the resulting increment of the flutter

speeds  $V_j$  ( $j \neq 1$ ):

$$\Delta V_j = \frac{W_1}{[1] \{\partial V_1 / \partial m_i\}} \left[ \frac{\partial V_1}{\partial m_i} \right] \left\{ \frac{\partial V_j}{\partial m_i} \right\} - B \left[ \frac{1}{\partial V_1 / \partial m_i} \right] \left\{ \frac{\partial V_j}{\partial m_i} \right\} \quad (5.36)$$

If all  $\Delta V_j \geq 0$  the optimization can be treated as a one-flutter-speed-constraint mass minimization. If there is one flutter speed, say  $V_2$ , for which  $\Delta V_2 < 0$  there is a possibility that the resizing column of

equation 5.20 reduces  $V_2$  below  $V_{R2}$ , and a condition for  $W_1$  can be established by equating  $\Delta V_2$  to its largest allowable negative value:

$$V_{R2} - V_2 .$$

$$V_2 = V_{R2} - V_2 = \frac{W_1}{[1] \left\{ \frac{\partial V_1}{\partial m_i} \right\}} \left[ \frac{\partial V_1}{\partial m_i} \right] \left\{ \frac{\partial V_2}{\partial m_i} \right\} - B \left[ \frac{1}{\frac{\partial V_1}{\partial m_i}} \right] \left\{ \frac{\partial V_2}{\partial m_i} \right\} \quad (5.37)$$

Since  $B$  is an implicit, non-linear function of  $W_1$  (equation 5.24), an iterative process, such as a one-dimensional Regula Falsi approach, should be used to determine  $W_1$  from equation 5.37. If  $W_1$  satisfying equation 5.37 is smaller than the  $W_1$  that follows from the step-size criterion associated with equations 5.20, 5.21 and 5.24 the former defines the value of  $W_1$  to be used for the first resizing. Otherwise the resizing is based on the criterion used in the case of one active flutter speed constraint.

If  $W_1$  as follows from equation 5.37 defines the step size, by definition  $V_1 = V_{R1}$  and  $V_2 = V_{R2}$ . For further weight reduction, while keeping  $V_1$  and  $V_2$  constant, the following may be considered.

In accordance with the basic resizing concept expressed in Section 5.3.2: add where it does the most good and subtract where it does the least harm, and taking equation 5.20 as a model, the following resizing column can be formed:

$$\{\Delta m_i\} = A \left\{ \frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i} \right\} - B \left\{ \frac{1}{\frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i}} \right\} \quad (5.38)$$

The requirements  $\Delta V_1 = 0$  and  $\Delta V_2 = 0$  lead to the following two complex determinantal equations which comprise four equations with the four unknowns: two reduced frequencies  $k_1$  and  $k_2$ , and  $A$  and  $B$ .

$$D \left( g, \gamma, k_1, V_{R1}, A \left\{ \frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i} \right\}, B \left\{ \frac{1}{\frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i}} \right\} \right) = 0 \quad (5.39)$$

$$D \left( g, \gamma, k_2, V_{R2}, A \left\{ \frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i} \right\}, B \left\{ \frac{1}{\frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i}} \right\} \right) = 0 \quad (5.40)$$

Although equations 5.39 and 5.40 suggest unique solutions for A and B, and the k values associated with  $V_{R1}$  and  $V_{R2}$ , these solutions may be complex values and thus without physical meaning.

It is noted that the two complex equations 5.39 and 5.40 with the four unknowns A, B,  $k_1$  and  $k_2$  can only be formulated because of the non-linear character of the relationships between the flutter speeds and the

design variables. Keeping  $\Delta V_1 = 0$  and  $\Delta V_2 = 0$  on a linear basis leads to:

$$\left[ \frac{\partial V_1}{\partial m_i} \right] \left( A \left\{ \frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i} \right\} - B \left\{ \frac{1}{\frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i}} \right\} \right) = 0 \quad (5.41)$$

$$\left[ \frac{\partial V_2}{\partial m_i} \right] \left( A \left\{ \frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i} \right\} - B \left\{ \frac{1}{\frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i}} \right\} \right) = 0 \quad (5.42)$$

which are two homogeneous equations with essentially one unknown: A/B, and in general, no solution for A and B separately.

It is also noted that, if real values of A and B are found, the distribution of  $\{\Delta m_i\}$  as well as its magnitude is defined. This may well lead to convergence difficulties.

A more promising approach is to select three linearly independent columns, such as shown in equation 5.43 or equation 5.44.

$$\{\Delta m_i\} = A \left\{ \frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i} \right\} - B \left\{ \frac{1}{\frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i}} \right\} + C \{1\} \quad (5.43)$$

$$\{\Delta m_i\} = A \left\{ \frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i} \right\} - B \left\{ \frac{1}{\frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i}} \right\} - C \left\{ \frac{1}{\frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i}} \right\} \quad (5.44)$$

A value of  $A$ , defining stepsize, can be selected, and equations equivalent to equations 5.41 and 5.42 make it possible to relate  $B$  and  $C$  to  $A$ . Since equations 5.41 and 5.42 are linear,  $V_1$  and  $V_2$  are not maintained exactly at  $V_R$ .

#### 5.4.3.3 The Distribution $\left\{ \frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i} \right\}$

Support for the use of the distribution  $\left\{ \frac{\partial V_1}{\partial m_i} + \frac{\partial V_2}{\partial m_i} \right\}$  as a guide for

generating a resizing column is found in the feasible directions procedure of Reference 16. It has been shown (see discussion in Reference 1, Section 6.5) that with the procedure suggested in Reference 16 (push-off factor  $\theta = 1$ ) the distribution of the resizing column can be determined from the following equations:

$$\begin{array}{c} \text{smallest} \\ \downarrow \\ \left\{ \begin{array}{c} \frac{\partial V_1}{\partial m_1} + \frac{\partial V_2}{\partial m_1} \\ \frac{\partial V_1}{\partial m_2} + \frac{\partial V_2}{\partial m_2} \\ \vdots \\ \frac{\partial V_1}{\partial m_n} + \frac{\partial V_2}{\partial m_n} \end{array} \right\} \rightarrow \{S\} = \left\{ \begin{array}{c} -1 \\ -1 \\ \vdots \\ S_k \\ S_{k+1} \\ \vdots \\ 1 \\ 1 \end{array} \right\} \\ \uparrow \\ \text{largest} \end{array} \quad (5.45)$$

$$[S] \left\{ \left\{ \frac{\partial V_1}{\partial m_i} \right\} + \{1\} \right\} = 0 \quad (5.46)$$

$$[S] \left\{ \left\{ \frac{\partial V_2}{\partial m_i} \right\} + \{1\} \right\} = 0 \quad (5.47)$$

The values of the elements  $S_k$  and  $S_{k+1}$  follow from the two equations with two unknowns: 5.46 and 5.47.

Thus design variables for which  $\frac{\partial V_1}{\partial m} + \frac{\partial V_2}{\partial m}$  is large are increased; those for which  $\frac{\partial V_1}{\partial m} + \frac{\partial V_2}{\partial m}$  is small are decreased.

Numerical results thus obtained were in exact agreement with results obtained using the Simplex method as suggested in Reference 16

### 5.5 Optimality Condition, Ballasting and Stiffening

The optimality condition for one flutter speed:

$$\frac{\partial V}{\partial m_1} = \frac{\partial V}{\partial m_2} = \frac{\partial V}{\partial m_3} = \dots \quad (5.48)$$

for free design variables is discussed in connection with practical considerations regarding ballast and stiffness design variables.

The literature pays little or no explicit attention to ballast (dead weight) as a design variable. The reason for this omission is understandable: any method of optimization that can handle design variables representing related stiffness and mass changes can handle a design variable representing a mass change only.

The need to consider mass ballast as a design variable is an additional argument in favor of using mass as the dimension of design variables.

It should be noted that a stiffness change with zero associated mass change cannot be represented by mass units. Such a stiffness change, however, must have some other penalty such as drag or production cost, and the objective function to be minimized cannot be the sum of structural and ballast mass.

Inclusion of design variables representing ballast obviously increases the number of design variables. Although ballasting design variables are less costly in term of computer time and problem formulation than stiffness design variables, an indiscriminate increase is an unwelcome burden. In Reference 1 it is suggested that it may be convenient to first optimize the structure using stiffness design variables and their associated mass only and then to determine whether any mass change by itself is more efficient than the stiffness changes in raising the flutter speed. Alternately, the values of flutter speed derivatives with respect to ballast mass, determined at a stiffness optimum, may be used in an attempt to further lower the total mass. The following procedure could be followed.

Let  $m_\sigma$  ( $\sigma = 1, 2, \dots, n_\sigma$ ) represent stiffness design variables and  $m_\beta$  ( $\beta = n_\sigma + 1, n_\sigma + 2, \dots, n$ ) ballast design variables. Then, after stiffness optimization:

$$\frac{\partial V}{\partial m_1} = \frac{\partial V}{\partial m_2} = \dots = \frac{\partial V}{\partial m_\sigma} = \dots = \frac{\partial V}{\partial m_{n_\sigma}} \quad (5.49)$$

for those  $m_\sigma$  not constrained by a sizing limit. Any  $m_\beta$  for which

$$\frac{\partial V}{\partial m_\beta} \leq \frac{\partial V}{\partial m_\sigma} \quad (5.50)$$

can not be used to reduce the total mass if it is assumed that initially there is no ballast present. This follows from the fact that equation 5.50 implies that the total mass can be reduced if  $m_\beta$  can be reduced and  $m_\sigma$  increased. If, however, there is a ballast configuration for which

$$\frac{\partial V}{\partial m_\beta} > \frac{\partial V}{\partial m_\sigma} \quad (5.51)$$

then a mass reduction by ballasting is possible.

Thus, after an optimization in which only stiffness design variables are used, a new optimization may be initiated that includes all stiffness variables and those ballasting design variables that satisfy equation 5.51.

Although sizing limits are usually thought of as lower limits, upper limits can also occur. Ballasting, for instance, has an upper limit defined by space available and specific weight of acceptable ballasting material. Even structural members may have upper limit constraints; for instance in areas where control surface actuators must be housed in a wing box. Thus the optimality condition of equation 5.48, when taking note of the thought expressed by equation 5.51 can be generalized as follows:

Let  $m_\ell$  be a design variable at its lower sizing limit,  $m_u$  a design variable at its upper limit, and  $m_i$  a design variable satisfying the condition of equation 5.48 then the generalized optimality condition is:

$$\frac{\partial V}{\partial m_i} = C \quad \frac{\partial V}{\partial m_\ell} < C \quad \frac{\partial V}{\partial m_u} > C \quad (5.52)$$

The conditions 5.52 can be used to check the result of an actual optimization process.

## 6. PRACTICAL CONSIDERATIONS RELATED TO OPTIMIZATION WITH FLUTTER CONSTRAINTS

Concurrent with the present study the Lockheed-California Company conducted work on contract NAS1-12288 with NASA, Langley Research Center (Reference 17). Contract NAS1-12288 is a design study for a supersonic transport with an arrow wing planform (Figure 6-1). During Task I of that contract three design concepts were evaluated, which included the determination of the weight penalty for satisfying the flutter constraints. On the basis of the results of Task I a design concept for Task II was chosen and further refined. One of the investigators in the present study participated in the weight minimization part of the arrow wing contract, with benefits for both studies.

Of importance to the present study is that the arrow wing contract provided a realistic design environment for flutter optimization.

The approach to flutter optimization consisted of a judicious choice of the most critical flutter condition followed by the actual optimization by means of an interactive Computer Graphics program, using the method of Incremented Flutter Analysis to determine the sensitivity of the Flutter speed to design variables.

The practical experience gained can be demonstrated by examining the significant findings during the Task I design of the arrow wing based on the chordwise stiffened design concept.

In the following the structural model and the flutter analyses of the original strength design are discussed. The structural regions delineating design variables are defined and the non-linear relations between elements of the stiffness matrix and the design variables are demonstrated. Finally the design experience as related to the number of modes used in the flutter optimization and the modal updating is presented.

### 6.1 Structural Model and Vibration Analysis

The Task I structural model of the wing is symmetric about a midplane parallel to the  $x$ - $y$  plane. Vertical deflections of the nodes on top and bottom surface of the wing are assumed equal and in the same direction, defining the overall deflection of the wing. Deflections in the horizontal plane on top and bottom surface are assumed equal and opposite and thus there are no elastic deflections parallel to the  $x$ - $y$  plane in the mid-plane of the wing. At each node the structural degrees of freedom are translations in the directions of the three axes. The fuselage is represented by a simple beam.

The total number of structural degrees of freedom per side is 1042. For symmetric modes 250 of these are translations in the  $z$  direction and, on



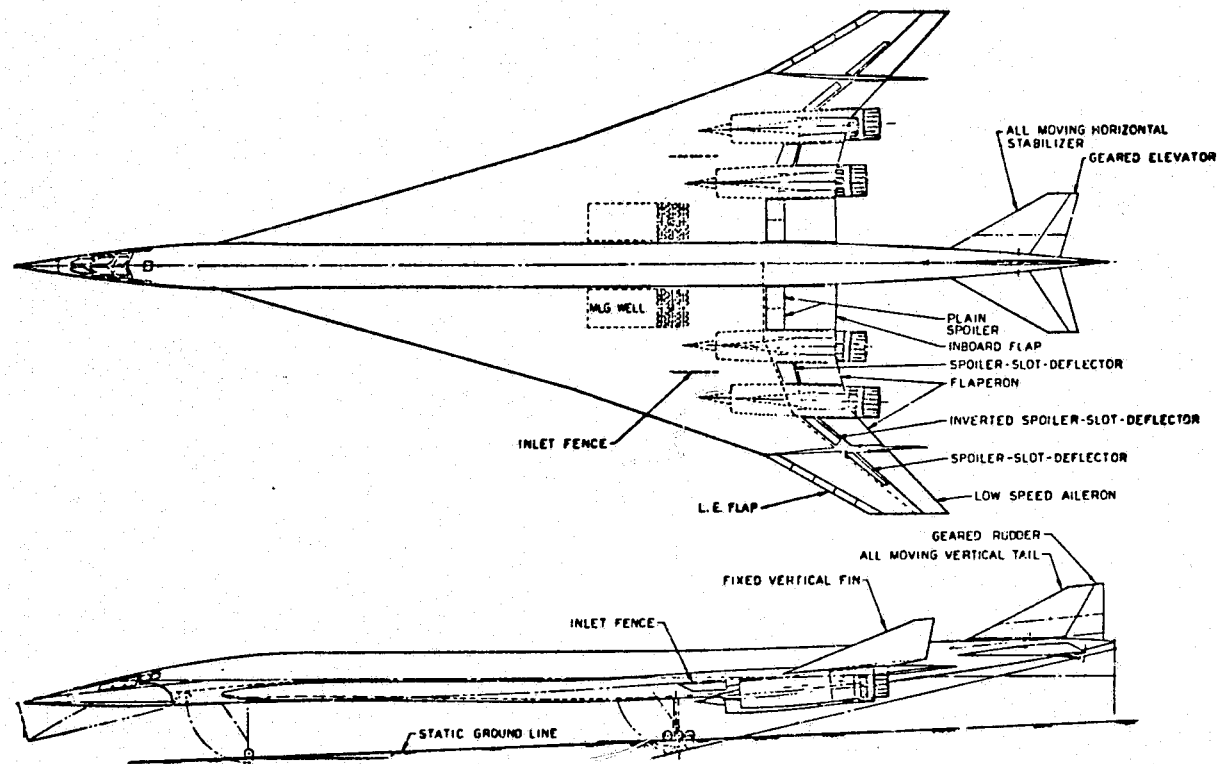


Figure 6-1: Arrow Wing Supersonic Transport.

the fin, 14 are translations in the y direction, describing the overall motion of the lifting surfaces. For antisymmetric modes 25 z translations on the fuselage are replaced by 25 y translations. The vibration and flutter equations are formed on the basis of 188 degrees of freedom for symmetric analyses (Figure 6-2) and 178 degrees of freedom for antisymmetric analyses. Consistent inertia and aerodynamics matrices are based on the more than 250 out-of-plane translations. A static reduction is used to reduce the structural degrees of freedom from 1042 to 188 (178 antisymmetric). A corresponding reduction is used for the inertia and aerodynamic matrices (Guyan reduction, Reference 18).

Vibration analyses are conducted to determine the lowest 50 natural frequencies and mode shapes. Table 6-1 summarizes the vibration characteristics of the initial, strength designed, configuration.

## 6.2 Aerodynamics and Flutter Analysis

Doublet lattice theory (Reference 9) is used to form 264th order aerodynamics matrices for  $M = 0.6$  and  $M = 0.9$ , using 233 aerodynamic boxes per side on the wing and 15 on the fin. They are subsequently reduced to 188th order by the Guyan reduction method (Reference 18).

The degrees of freedom in the flutter analysis are reduced by post and pre multiplying the flutter matrix by a modal matrix and its transpose. Twenty vibration modes are used in the flutter survey of the initial configuration.

Sample results of the flutter analysis are shown in Figures 6-3 through 6-6.

The use of twenty modes was based on engineering judgment in reaching a compromise between accuracy and cost. Cost considerations included not only the direct cost per flutter analysis, but also the cost of assuring that the number of modes chosen, if low, is adequate.

An example of the sensitivity of the flutter characteristics to number of modes used is presented in Figure 6-7. For this condition it appears 15 modes would have given acceptable results for the flutter analysis. Regardless of this answer, however, twenty modes was considered the absolute minimum number of modes required for a meaningful optimization. The effect of increasing the number of modes on one flutter speed is shown in Figure 6-8.

To increase the understanding of the nature of the flutter modes, modal participation coefficients were plotted to make visible the amount of participation of each of the twenty natural vibration modes in in-flight modes of interest as a function of speed. Examination of these plots revealed that the flutter mode labeled bending and torsion and the one labeled hump mode

Figure 6-2: Symmetric Degrees of Freedom for Vibration Analysis.

MODE DESCRIPTION	MODAL FREQUENCY - HERTZ			
	SYMMETRIC		ANTISYMMETRIC	
	OWE	FFFP	OWE	FFFP
WING 1ST BENDING	1.009	0.933	1.228	0.908
FUSELAGE 1ST BENDING	1.381	1.206	1.998	1.949
ENGINE PITCH IN PHASE	1.641	1.627	1.514	1.457
ENGINE PITCH OUT OF PHASE	1.817	1.815	1.821	1.805
FUSELAGE 2ND BENDING	2.784	2.261	3.370	3.056
WING 1ST TORSION	3.288	3.104	3.034	3.056

Operating Weight Empty (OWE) = 321,000 LBS

Full Fuel Full Payload (FFFP) = 750,000 LBS

Table 6-1: Summary of Vibration Characteristics; Initial Design.

SYMMETRIC FLUTTER ANALYSIS - CHORDWISE STIFFENED ARRANGEMENT  
MACH NO. = 0.6  
WEIGHT = 321,000 LBS

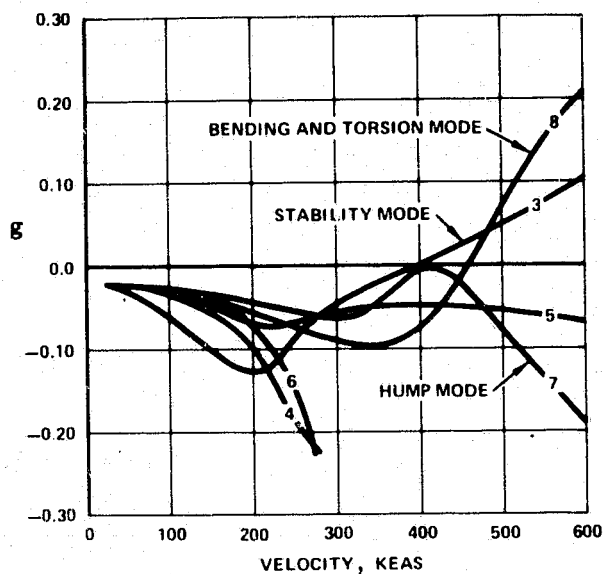
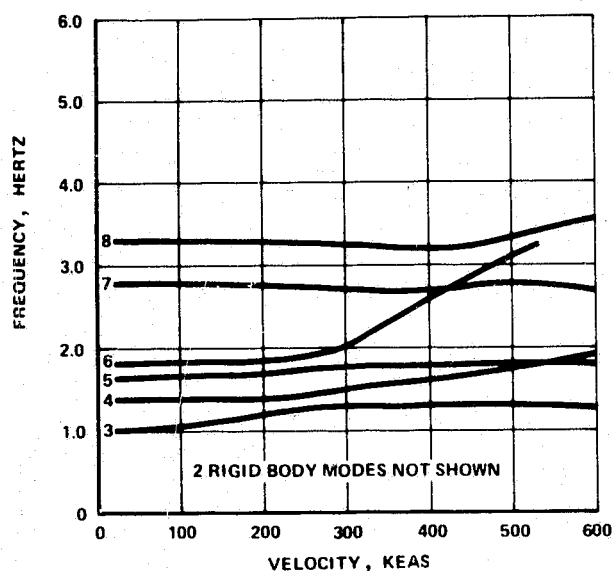


Figure 6-3: Symmetric Flutter Analysis - Mach 0.6 - Operating Weight Empty.

SYMMETRIC FLUTTER ANALYSIS - CHORDWISE STIFFENED ARRANGEMENT  
MACH NO. = 0.9  
WEIGHT = 750,000 LBS

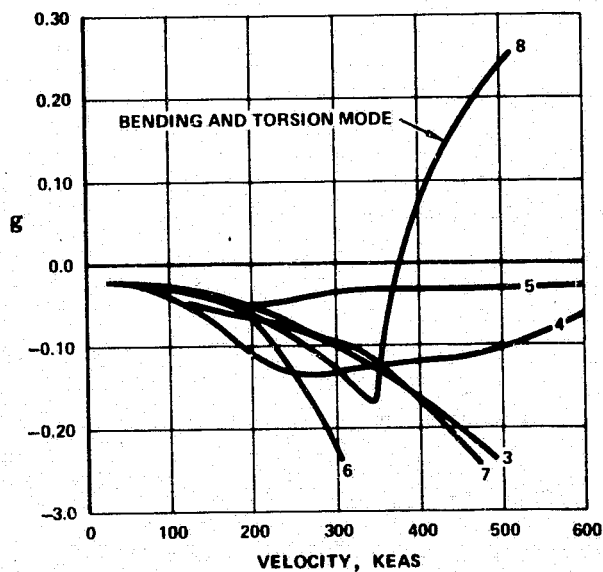
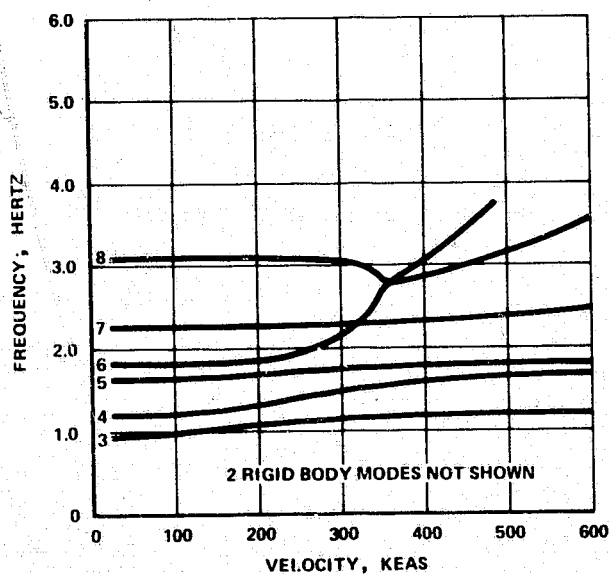


Figure 6-4: Symmetric Flutter Analysis - Mach 0.9 - Full Fuel Full Payload.

**ANTISYMMETRIC FLUTTER ANALYSIS - CHORDWISE STIFFENED ARRANGEMENT**  
**MACH NO. = 0.9**  
**WEIGHT = 321,000 LBS.**

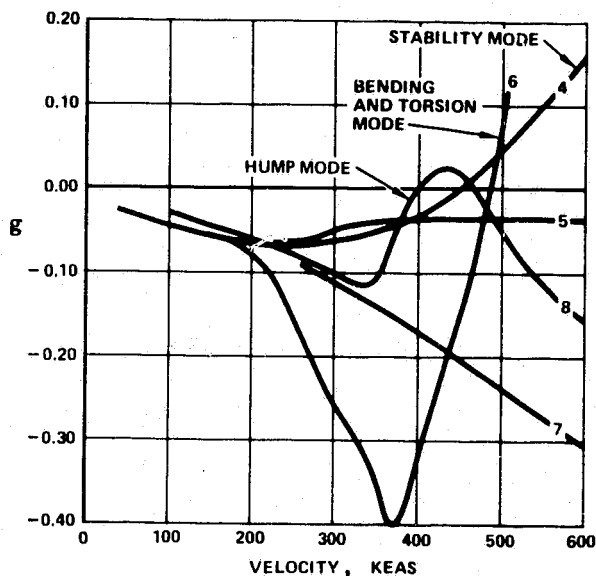
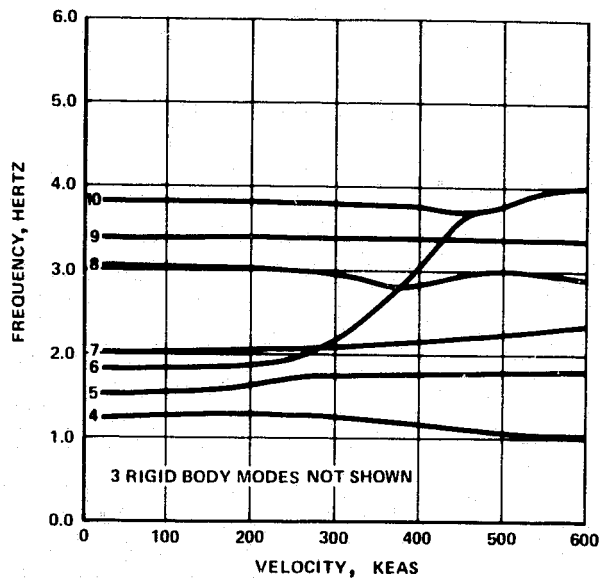


Figure 6-5: Antisymmetric Flutter Analysis - Mach 0.9 - Operating Weight Empty.

**ANTISYMMETRIC FLUTTER ANALYSIS - CHORDWISE STIFFENED ARRANGEMENT**  
**MACH NO. = 0.9**  
**WEIGHT = 750,000 LBS.**

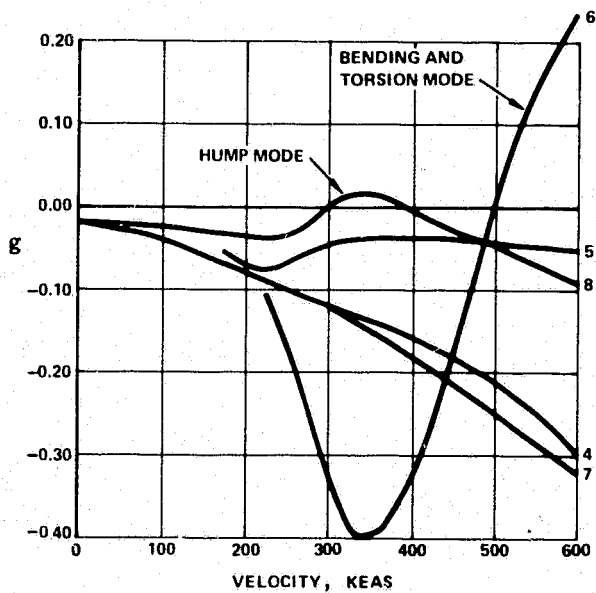
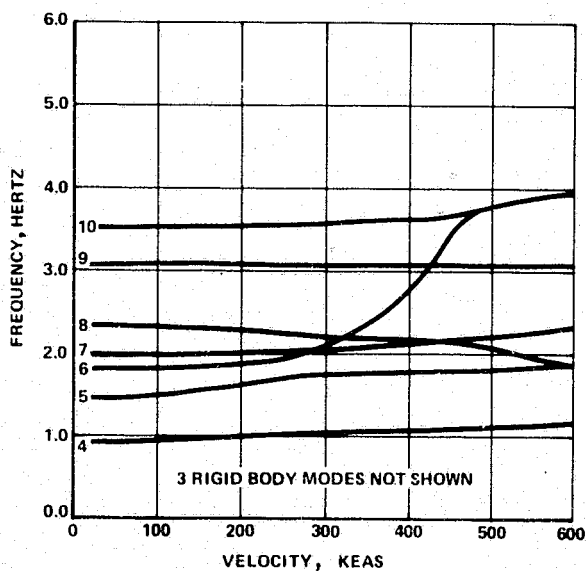


Figure 6-6: Antisymmetric Flutter Analysis - Mach 0.9 - Full Fuel Full Payload.

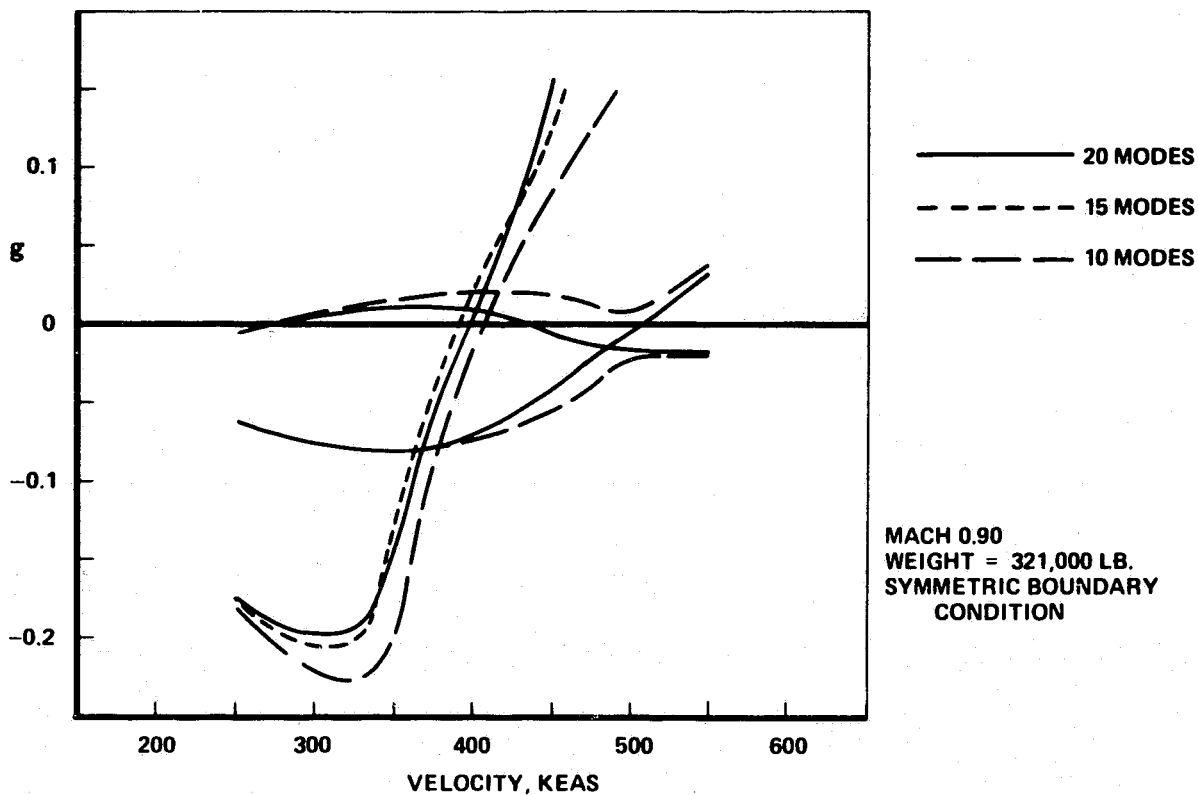


Figure 6-7: The Effects of Number of Modes on the Flutter Solution.

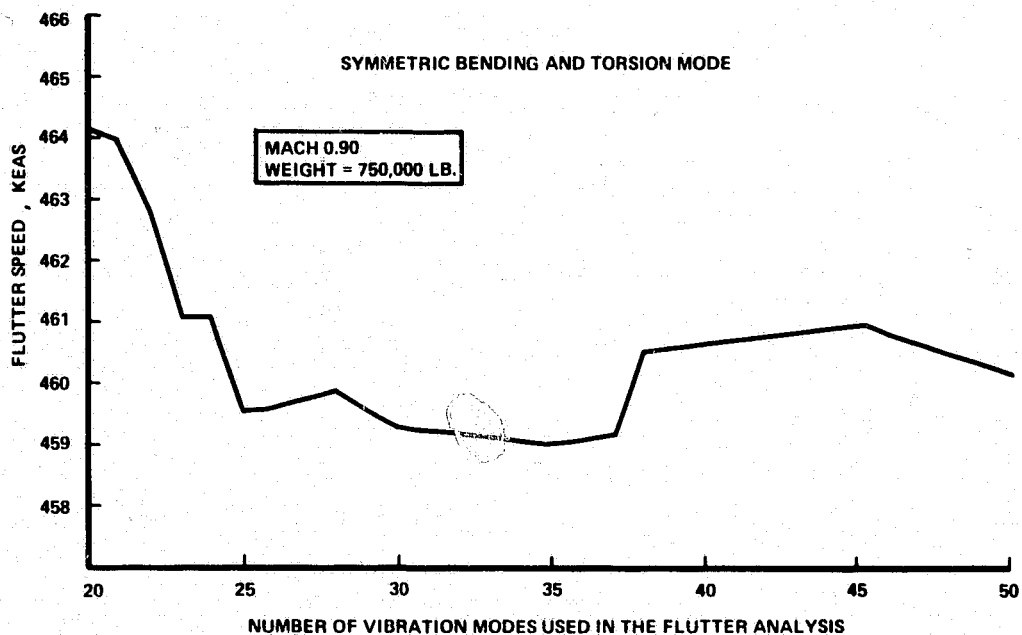


Figure 6-8: Flutter Speed Variation With Number of Modes.

probably could benefit from wing stiffening. However, the mode marked stability mode proved to be composed of the first elastic mode and rigid body modes and it was not expected to gain significantly in flutter speed by wing stiffening. In view of its strong dependency on the rigid body stability characteristics, which during Task I were not well defined, this mode was not considered during the Task I optimization.

Additional insight was gained by a flutter analysis of a configuration rigidized inboard from butt line 470 (Figure 6-9). The flutter mode with a flutter speed of 418 KEAS is very similar in character to the bending and torsion mode of Figure 6-4 with a flutter speed of 379 KEAS.

On the basis of the results shown, and additional flutter results for other combinations of Mach number and weight, the symmetric bending and torsion flutter mode at  $M = 0.9$  and 750,000 pounds was chosen for the optimization task.

### 6.3 Optimization

As indicated in the previous section the full fuel, full payload, 750,000 pounds configuration, chordwise stiffened concept, at  $M = 0.9$  was chosen for the optimization procedure. The flutter speed of the symmetric bending and torsion mode of the initial configuration was 379 KEAS. The flutter optimization task was to resize the structure to increase the flutter speed to  $1.2 V_D$  (468 KEAS) with a minimum weight penalty.

To reduce the optimization task to a manageable size relative to available resources the wing planform was divided into eight regions (Figure 6-10). For each region a design variable related to the torsional stiffness and a design variable related to the bending stiffness were defined. Increasing the torsional stiffness was accomplished by increasing skin thickness and beam web thickness, which resulted in increased bending stiffness as well.

Increasing the bending stiffness was accomplished by increasing the beam cap areas and resulted in a small increase in torsional stiffness due to differential bending. Design regions 9 and 10 are the bending stiffnesses of the engine supports. They were not varied during the Task I optimization procedure.

The static reduction of the stiffness matrix from 1042 to 188 degrees of freedom led to non-linear relations between elements of the 188th order stiffness matrix and the design variables  $\beta$ , as illustrated in a few examples in Figure 6-11. The stiffness matrix, as a function of all design variables  $\beta$ , was approximated by:

$$[K(\beta_i)] \approx [K(0)] + \sum_{i=1}^{16} \left( [A_i] + \beta_i [B_i] + \beta_i^2 [C_i] \right) \quad (6.1)$$

MACH NO. = 0.9  
WEIGHT = 750,000 LBS.

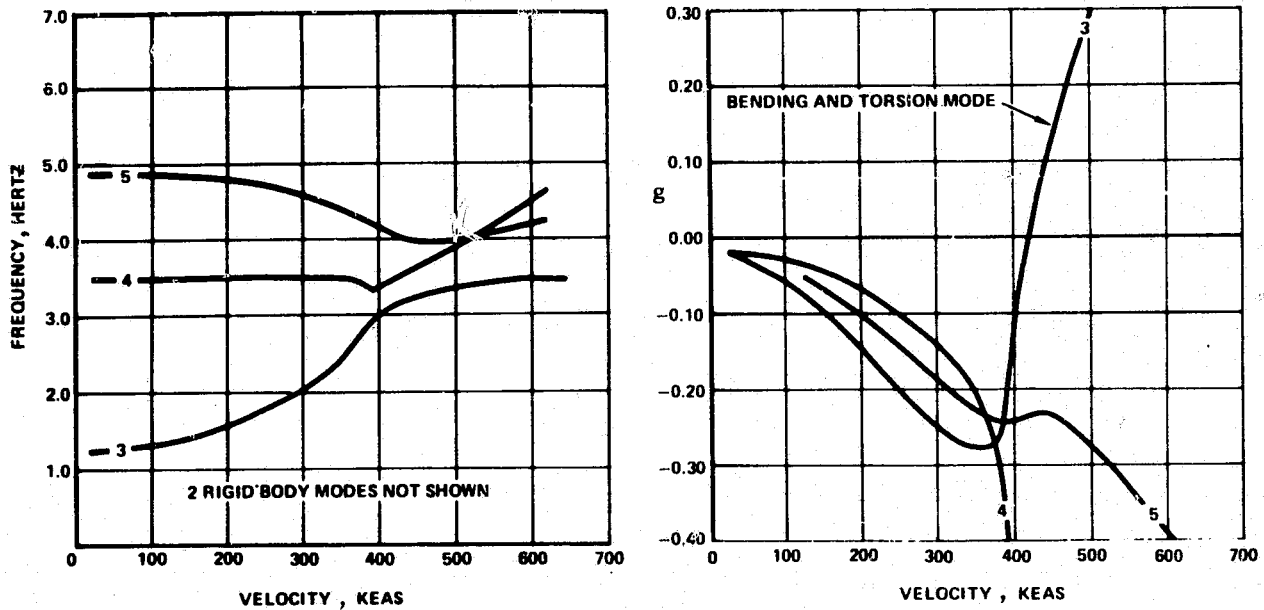


Figure 6-9: Symmetric Flutter Analysis - Mach 0.9 - Rigidized Inner Wing.

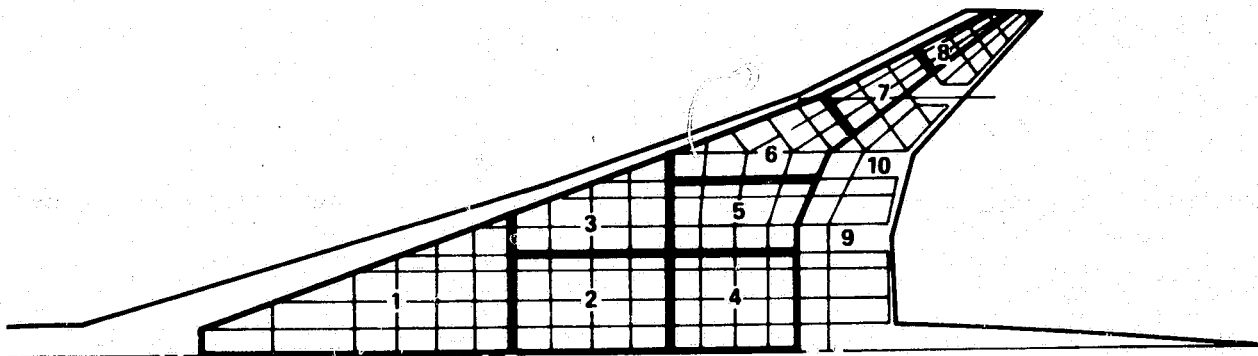
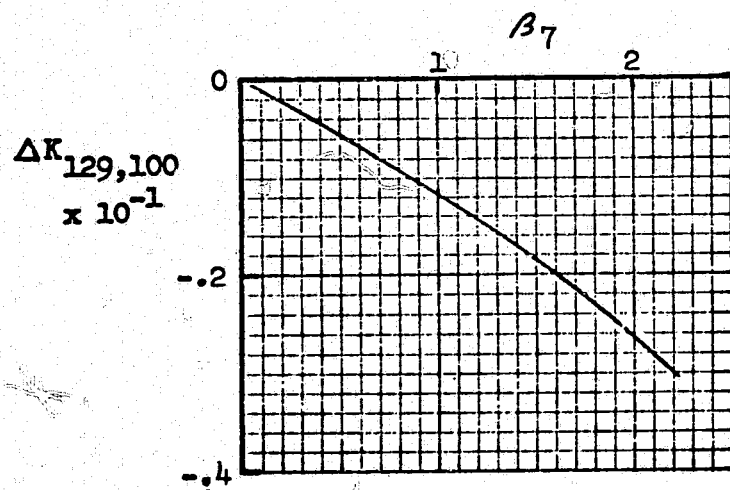
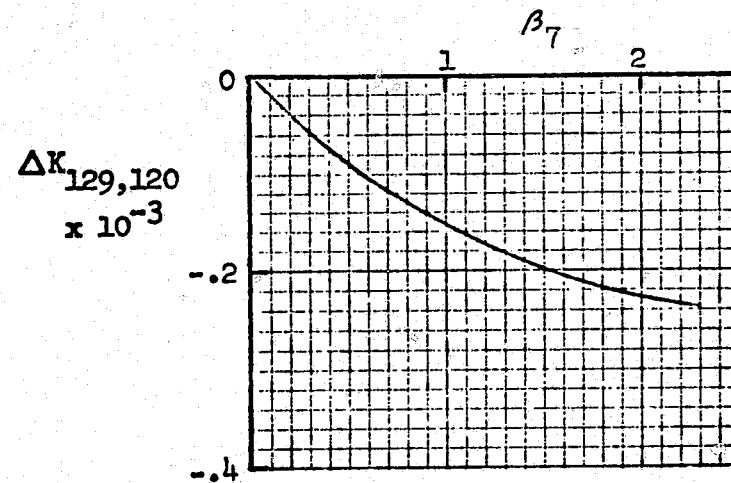


Figure 6-10: Design Regions for Flutter Optimization.





$\beta_1 = 0, 1 \neq 7$



$\beta = 0$  is initial sizing.  
Matrix element numbers  
correspond to Figure 6-2.

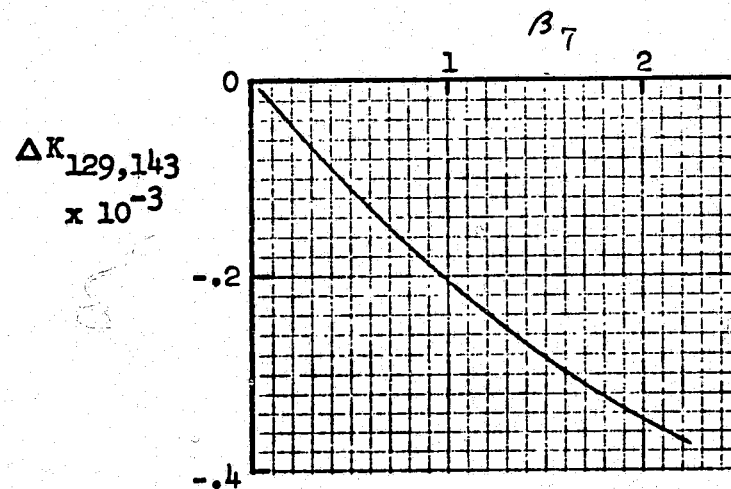
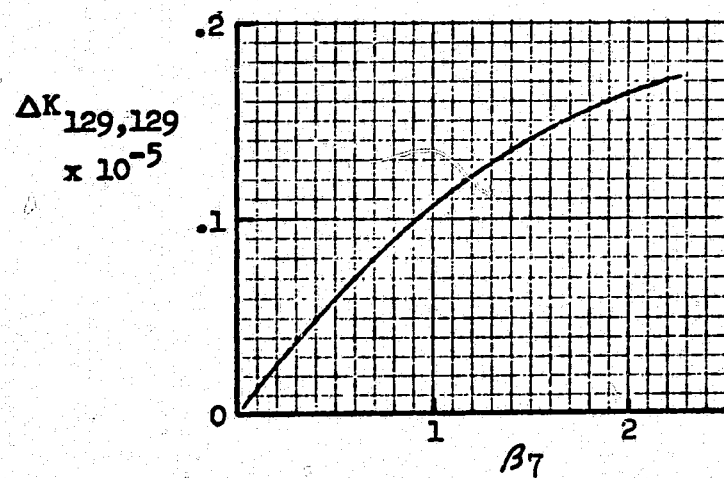


Figure 6-11: Elements of the Stiffness Matrix as a Function of a Design Variable.

It was found that interaction between the design variables was minor so that the  $\beta_i \beta_j$  terms were neglected in the approximation.

The optimization technique used is an interactive Computer Graphics program, using the method of Incremented Flutter Analysis, in which successive combinations of design variables with decreasing total mass are generated while keeping the flutter speed exactly at the desired value. The effectiveness of design variables in increasing the flutter speed per unit mass is determined similar to the procedure described in Section 3.2. On the basis of the current values for the effectiveness the next resizing is determined by engineering judgment, after which new sensitivities are determined. The process is continued until the sensitivities have become approximately equal and no further weight reduction is possible. If the number of design variables that play a role is small, say smaller than 20, this technique is very efficient in terms of resources required and elapsed time.

The basic computer program in this optimization technique is executed with fixed modalization, although it can be interspersed at will with modal updating. The increase of the flutter speed to the desired value can also be done in several steps. The practical experience gained while following one particular sequence of steps is discussed. Reference is made to Table 6-2.

The initial configuration ( $S_0$ ) and its natural vibration modes obtained from a 188th order vibration analysis ( $VM_0$ ) had a flutter speed of 379 KEAS. The flutter equation was modalized with the lowest 20 of these modes. An optimization using fixed modes was performed and it was found that to reach a flutter speed of 468 KEAS, 1105 pounds per side had to be added. This first "optimized" solution ( $S_1$ ) consisted of 682.5 pounds of structure for the torsion design variable and 422.5 pounds for the bending design variable in Region 8 (Figure 6-10). For the configuration thus defined, a new vibration analysis ( $VM_1$ ) using 188 degrees of freedom was conducted followed by a 20 mode flutter analysis. The resultant flutter speed was 443 KEAS, 25 KEAS less than the desired value. Re-analyzing this structure using the original 20 vibration modes led to the desired flutter speed of 468 KEAS, indicating that it was the use of fixed modes that caused the optimization process to result in a deficient structure.

Using the newly computed vibration modes ( $VM_1$ ) in a fixed mode optimization process indicated the need of an additional 492 pounds per side (for a total of 1597 pounds per side added) to reach a flutter speed of 468 KEAS. This second solution ( $S_2$ ) had 1470 pounds of bending material and only 126.7 pounds of torsion material. Thus, it appears that for twenty modes this process of fully optimizing and then performing a vibration analysis may not converge on the optimum distribution among the several design variables. This approach was considered unpromising and has not been pursued further. No updated flutter analysis of configuration  $S_2$  was performed.

Configu- ration	Vibration Analysis 188th Order	Flutter Analysis			Optimization						
					Modes Used	Order	Resultant Configuration				
		Flutter Speed KEAS	Modes Used	Order			Identi- fication	Weight Added to Base Configuration, lbs per Side			
								Region	Torsion	Bending	Total
S <sub>0</sub> (base)	VM <sub>0</sub>	VM <sub>0</sub>	20	379	VM <sub>0</sub>	20	S <sub>1</sub>	8	682.5	422.5	1105
S <sub>1</sub>	VM <sub>1</sub>	VM <sub>1</sub>	20	443	VM <sub>1</sub>	20	S <sub>2</sub>	8	126.7	1470.0	1596.7
		VM <sub>0</sub>	20	468							
		VM <sub>0</sub>	50	445	VM <sub>0</sub>	50	S <sub>3</sub>	7 8	82.0 861.4	82.0 778.4	1721.8
S <sub>3</sub>	VM <sub>3</sub>	VM <sub>3</sub>	50	460	VM <sub>3</sub>	50	S <sub>4</sub>	7 8	490.0 704.0	490.0 755.0	1949.0

Table 6-2: Arrow Wing Optimization

To maximize the possibility of obtaining a meaningful optimized configuration the maximum number of modes, 50, that could be accommodated on the interactive system was used. First configuration  $S_1$  was analyzed, using 50 of the  $VM_0$  modes. This resulted in a flutter speed of 445 KEAS, very close to the 443 KEAS found with 20  $VM_1$  modes. Using the 50  $VM_0$  modes another optimization was performed, leading to  $S_3$  with 861.4 pounds in torsion and 778.4 pounds in bending added in Region 8 and 82 pounds in bending in Region 7. A 188th order vibration analysis of  $S_3$  resulted in the modes  $VM_3$ , 50 of which led to a flutter speed of 460 KEAS. A second 50 fixed modes optimization was performed using the  $VM_3$  modes. This led to the configuration  $S_4$ : 704 pounds in torsion and 755 pounds in bending in Region 8 and 490 pounds in bending in Region 7, for a total weight penalty of 1949 pounds per side.

It was judged that remodelizing the  $S_4$  configuration would lead to a 50 modes flutter speed acceptably close to 468 KEAS and that the modes would change little, indicating that a minimum weight was approximated.

#### 6.4 General Conclusions

Some of the practical experience with flutter optimization in a realistic design environment gained during the Contract NAS1-12288 flutter task is illustrated in the preceding section. Additional experience was gained during Task I with two other structural design concepts, and during Task II with a mixed chordwise stiffened-monocoque design. Each optimization was done for a 750,000 pound configuration at  $M = 0.9$ . During Task II an optimization at  $M = 1.85$  was added since the  $M = 0.9$  optimization led to a flutter speed less than  $1.2 V_D$  at  $M = 1.85$  (Reference 19).

The practical experience gained by the present investigators during the entire arrow wing contract can be summarized in the following conclusions.

1. An airplane with relatively complicated in-flight modal characteristics was optimized considering only one flutter speed at a time.
2. The optimization of the arrow wing configurations was accomplished with the help of man-in-the-loop techniques. The resulting restrictions regarding the number of design variables and structural and modal updating emphasize the need for more powerful optimization programs.
3. Reduction of the stiffness matrix and the associated non-linearity leads to the structural analysis being a dominant part of the cost of optimization. Future optimization studies should aim at reducing this cost. Such cost reduction may be found in using a simple structural model or in using approximate methods. (See also discussion in Reference 1.)

4. Frequent updating of the vibration modes assists in speeding up the attaining of a converged optimum design. An error in judgment, underestimating the frequency required, however, will be caught when the final, supposedly optimized design is remodelized and a check flutter analysis performed. Such a check should be a routine procedure. The use of too low a number of vibration modes, or insufficient design regions, however, is not "automatically" checked for accuracy, but would require effort outside the mainstream of the optimization procedure.

## 7. DERIVATIVES WITH RESPECT TO DESIGN VARIABLES

The resizing of the structure in a structural optimization with flutter constraints is directly related to the sensitivity of the flutter speed with respect to the design variables. In 1947, van der Vooren (Reference 20) published a method of expressing the flutter speed and mode of a slightly perturbed configuration in terms of the flutter speed and mode of the original configuration and the perturbation. More recently, and directly in connection with flutter optimization work, Rudisill and Bhatia (Reference 6) have derived expressions for the derivative of the flutter speed with respect to design variables. A compact description of the method is included for completeness in Section 7.1.

Section 7.2 is based on work performed during this study. It presents a method of obtaining derivatives of the damping at a given speed with respect to design variables.

For both types of derivatives, the derivatives of the mass, stiffness and aerodynamics matrices are needed. Present finite element structural modelling technique leads to satisfactory structural representation for aeroelastic purposes while maintaining linear relationships between the mass and stiffness matrices and the design variables. Thus the mass matrix  $[M(m_i)]$  and the stiffness matrix  $[K(m_i)]$ , for arbitrary values of  $m_i$ , can be written as:

$$[M(m_i)] = [M_0] + \sum_{i=1}^n m_i [\Delta M_i] \quad (7.1)$$

and

$$[K(m_i)] = [K_0] + \sum_{i=1}^n m_i [\Delta K_i] \quad (7.2)$$

where  $[M_0]$  and  $[K_0]$  are base mass and stiffness matrices, respectively, and  $[\Delta M_i]$  and  $[\Delta K_i]$  are incremental matrices corresponding to a unit of  $m_i$ .

If the order of the stiffness matrix is too large for efficient repeated vibration analyses, as required in an optimization procedure, static reduction may be necessary; this may destroy the linear relationship shown in equation 7.2. However, as described in Reference 1, the derivative of the stiffness matrix for  $m_i = 0$  can be obtained from an expression identical to equation 7.2. For each design variable for which  $m_i \neq 0$ , new matrices  $[K_0]$  and  $[\Delta K_i]$  must be computed to determine the derivatives.

The derivative of the aerodynamics matrix can be obtained by finite difference techniques or by the methods discussed in Section 5 of Reference 1.

### 7.1 Derivatives of the Flutter Speed

The flutter speed,  $V$ , and the associated reduced frequency,  $k = \frac{\omega c}{V}$ , are obtained from the characteristic flutter equation:

$$\left[ \frac{-1}{c^2} [M] k^2 + \frac{1+ig}{v^2} [K] - \frac{1}{2} \rho [A(ik)] \right] \{q\} = 0 \quad (7.3)$$

where  $g$  is the structural damping

Since at flutter  $\gamma=0$ , equation 7.3 is valid for the  $k$ -method approach as well as for the  $p$ - $k$  method approach.

The matrix  $\{q\}$  is the modal column associated with the flutter solution. For the same determinant, but associated with the transposed matrix problem, there exists the corresponding column  $\{r\}$ . Thus:

$$[r] \left[ -\frac{1}{c^2} [M] k^2 + \frac{1+ig}{v^2} [K] - \frac{1}{2} \rho [A(ik)] \right] = 0 \quad (7.4)$$

The derivative of equation 7.3 with respect to the design variable  $m$  is:

$$\left[ -\frac{1}{c^2} \left[ \frac{\partial M}{\partial m} \right] k^2 - \frac{2}{c^2} [M] k \frac{\partial k}{\partial m} - \frac{2(1+ig)}{v^3} \frac{\partial v}{\partial m} [K] + \frac{1+ig}{v^2} \left[ \frac{\partial K}{\partial m} \right] + \right. \\ \left. - \frac{1}{2} \rho \left[ \frac{\partial A}{\partial k} \right] \frac{\partial k}{\partial m} \{q\} + \left[ -\frac{1}{c^2} [M] k^2 + \frac{1+ig}{v^2} [K] - \frac{1}{2} \rho [A(ik)] \right] \left\{ \frac{\partial q}{\partial m} \right\} = 0 \quad (7.5)$$

Pre-multiplying 7.5 with  $[r]$  and invoking equation 7.4 leads to:

$$-\frac{1}{c^2} \left( r \frac{\partial M}{\partial m} q \right) k^2 - \frac{2}{c^2} (rMq) k \frac{\partial k}{\partial m} - \frac{2(1+ig)}{v^3} \frac{\partial v}{\partial m} (rKq) + \\ + \frac{1+ig}{v^2} \left( r \frac{\partial K}{\partial m} q \right) - \frac{1}{2} \rho \left( r \frac{\partial A}{\partial k} q \right) \frac{\partial k}{\partial m} = 0 \quad (7.6)$$

Equation 7.6 is a complex scalar equation. The scalar  $r \frac{\partial M}{\partial m} q$  stands for  $[r] \left[ \frac{\partial M}{\partial m} \right] \{q\}$ . Similarly the other triple products between parentheses stem from triple matrix products. In equation 7.6  $k$ ,  $v$ ,  $\{q\}$ , and  $[r]$  are known solutions of the equations 7.3 and 7.4. Equation 7.6 represents two linear equations from which the two unknowns  $\frac{\partial k}{\partial m}$  and  $\frac{\partial v}{\partial m}$  can readily be obtained.

It may be of interest to return to equation 7.5 and determine the derivative of the flutter mode:  $\left\{ \frac{\partial q}{\partial m} \right\}$ .

With  $\frac{\partial k}{\partial m}$  and  $\frac{\partial v}{\partial m}$  determined from equation 7.6,  $\left\{ \frac{\partial q}{\partial m} \right\}$  is the only unknown in equation 7.5. Since equation 7.3 defines only a distribution for  $\{q\}$ ,  $\{q\}$  can be normalized. Let  $\{q\}$  be normalized to the last element, which then, by definition, is a constant. It follows that the last element of  $\left\{ \frac{\partial q}{\partial m} \right\}$  is zero, which can be omitted if the last columns of the matrices pre-multiplying  $\left\{ \frac{\partial q}{\partial m} \right\}$  are omitted. Equation 7.5 then has one more equation

than there are unknowns  $\frac{\partial q}{\partial m}$  and, arbitrarily, the equation corresponding to the last row of all matrices is omitted.

If  $[M|]$  means the last column of  $[M]$  is omitted, and  $[\underline{M}]$  means the last row is omitted, the solution for  $\left\{\frac{\partial q}{\partial m}\right\}$  can be written as:

$$\left\{\frac{\partial q}{\partial m}\right\} = - \left[ -\frac{1}{c^2} [\underline{M}] k^2 + \frac{1+ig}{v^2} [\underline{K}] - \frac{1}{2} \rho [\underline{A(ik)}] \right]^{-1} \cdot \left[ -\frac{1}{c^2} \left[\frac{\partial M}{\partial m}\right] k^2 + \right. \\ \left. - \frac{2}{c^2} [\underline{M}] k \frac{\partial k}{\partial m} - \frac{2(1+ig)}{v^3} \frac{\partial v}{\partial m} [\underline{K}] + \frac{1+ig}{v^2} \left[\frac{\partial K}{\partial m}\right] - \frac{1}{2} \rho \left[\frac{\partial A}{\partial k}\right] \right] \{q\} \quad (7.7)$$

The complete  $\left\{\frac{\partial q}{\partial m}\right\}$  column is given by:

$$\left\{\frac{\partial q}{\partial m}\right\} = \begin{Bmatrix} \frac{\partial q}{\partial m} \\ 0 \end{Bmatrix} \quad (7.8)$$

The second derivative of the flutter speed is used in some methods of optimization that use a defined step size (Reference 1, Section 7). Two equations with the two unknowns  $\frac{\partial^2 k}{\partial m_i \partial m_j}$  and  $\frac{\partial^2 v}{\partial m_i \partial m_j}$  can be derived by assuming the derivatives in equation 7.5 are with respect to  $m_i$  and then differentiating with respect to  $m_j$ . Because of equation 7.1,  $\left[\frac{\partial^2 M}{\partial m_i \partial m_j}\right] = 0$ .

It can be shown that for  $m_i = 0$ ,  $\left[\frac{\partial^2 K}{\partial m_i \partial m_j}\right] = 0$ . The unknowns  $\frac{\partial^2 k}{\partial m_i \partial m_j}$  and

$\frac{\partial^2 v}{\partial m_i \partial m_j}$  can be expressed in terms of the originally known quantities and the quantities to be computed:  $\frac{\partial k}{\partial m_i}$ ,  $\frac{\partial k}{\partial m_j}$ ,  $\frac{\partial v}{\partial m_i}$ ,  $\frac{\partial v}{\partial m_j}$ ,  $\left\{\frac{\partial q}{\partial m_i}\right\}$  and  $\left\{\frac{\partial q}{\partial m_j}\right\}$ .



## 7.2 Derivatives of the Flutter Root

The flutter root,  $p$ , defines the frequency and damping of an in-flight mode at a given speed. It satisfies the characteristic equation of the flutter equation:

$$\left[ \frac{v^2}{c^2} [M] p^2 + (1+ig) [K] - \frac{1}{2} \rho v^2 [A(p)] \right] \{q\} = 0 \quad (7.9)$$

Let  $p = (\gamma + i)k$  be a root of equation 7.9 and  $\{q\}$  the associated modal column. The derivatives  $\frac{\partial \gamma}{\partial m}$  would be required in an optimization process in which there is a damping constraint. Damping may be a flutter constraint (as discussed in Reference 1, Section 7.3), but could be a constraint related to airplane stability characteristics.

$\frac{\partial \gamma}{\partial m}$  can be obtained from  $\frac{\partial p}{\partial m} = a + ib$ :

$$\frac{\partial \gamma}{\partial m} = \frac{a - \gamma b}{k} \quad (7.10)$$

$\frac{\partial p}{\partial m}$  is obtained from equation 7.9 as follows:

For a given Mach number and altitude  $v^2/c^2$  and  $\frac{1}{2} \rho v^2$  are constant. To simplify the notation  $\frac{v^2}{c^2} [M]$  is replaced by  $[\tilde{M}]$ ,  $(1+ig) [K]$  by  $[\tilde{K}]$  and  $\frac{1}{2} \rho v^2 [A(p)]$  by  $[\tilde{A}(p)]$ . Equation 7.9 then becomes:

$$\left[ [\tilde{M}] p^2 + [\tilde{K}] - [\tilde{A}(p)] \right] \{q\} = 0 \quad (7.11)$$

Differentiation of equation 7.11 with respect to the design variable  $m$  leads to:

$$\left[ \left[ \frac{\partial \tilde{M}}{\partial m} \right] p^2 + 2 [\tilde{M}] p \frac{\partial p}{\partial m} + \left[ \frac{\partial \tilde{K}}{\partial m} \right] - \left[ \frac{\partial \tilde{A}}{\partial p} \right] \frac{\partial p}{\partial m} \right] \{q\} +$$

$$+ \left[ [\tilde{M}] p^2 + [\tilde{K}] - [\tilde{A}(p)] \right] \left\{ \frac{\partial q}{\partial m} \right\} = 0$$

Pre-multiplication by  $[r]$ , where  $[r]$  satisfies equation 7.13 for the flutter root of interest, leads to the expression for  $\frac{\partial p}{\partial m}$  given in equation 7.14.

$$[r] \left[ [\tilde{M}] p^2 + [\tilde{K}] - [\tilde{A}(p)] \right] = 0 \quad (7.13)$$

$$\frac{\partial p}{\partial m} = \frac{[r] \left[ \left[ \frac{\partial \tilde{K}}{\partial m} \right] + p^2 \left[ \frac{\partial \tilde{M}}{\partial m} \right] \right] \{q\}}{[r] \left[ \left[ \frac{\partial \tilde{A}}{\partial p} \right] - 2p [\tilde{M}] \right] \{q\}} \quad (7.14)$$

Following a reasoning similar to that in Section 7.1, the derivative of the flutter mode can be derived:

$$\left\{ \frac{\partial q}{\partial m} \right\} = - \left[ [\tilde{M}] p^2 + [\tilde{K}] - [\tilde{A}(p)] \right]^{-1} \left[ \left[ \frac{\partial \tilde{M}}{\partial m} \right] p^2 + 2 [\tilde{M}] p \frac{\partial p}{\partial m} + \left[ \frac{\partial \tilde{K}}{\partial m} \right] - \left[ \frac{\partial \tilde{A}}{\partial p} \right] \frac{\partial p}{\partial m} \right] \{q\} \quad (7.15)$$

Assuming the derivatives in equation 7.12 are with respect to  $m_i$ , this equation can be differentiated with respect to  $m_j$  and an expression

for  $\frac{\partial^2 p}{\partial m_i \partial m_j}$  can be obtained, from which

$$\frac{\partial^2 p}{\partial m_i \partial m_j} = \frac{c - \gamma d}{k} - \frac{a_i b_j - 2 \gamma b_i b_j + a_j b_i}{k^2} \quad (7.16)$$

The  $a$ ,  $b$ ,  $c$  and  $d$  quantities in equation 7.16 are defined by:

$$\left. \begin{aligned} \frac{\partial p}{\partial m_i} &= a_i + i b_i & \frac{\partial p}{\partial m_j} &= a_j + i b_j & \frac{\partial^2 p}{\partial m_i \partial m_j} &= c + i d \end{aligned} \right\} \quad (7.17)$$

Since it is not certain whether acceptable approximations to  $[A(p)]$  can be derived from  $[A(ik)]$  given at discrete values  $k_l$ , an alternate approach to determining  $\frac{\partial \gamma}{\partial m}$  is presented for consideration.

In the p-k method (Reference 3) equation 7.11 is written as:

$$\left[ [\tilde{M}] p^2 + [\tilde{K}] - [\tilde{A}(ik)] \right] \{q\} = 0 \quad (7.18)$$

Differentiation gives:

$$\left[ \left[ \frac{\partial \tilde{M}}{\partial m} \right] p^2 + 2 [\tilde{M}] p \frac{\partial p}{\partial m} + \left[ \frac{\partial \tilde{K}}{\partial m} \right] - \left[ \frac{\partial \tilde{A}}{\partial k} \right] \frac{\partial k}{\partial m} \right] \{q\} + \left[ [\tilde{M}] p^2 + [\tilde{K}] - [\tilde{A}(ik)] \right] \left\{ \frac{\partial q}{\partial m} \right\} = 0 \quad (7.19)$$

$$\frac{\partial p}{\partial m} = \frac{\partial \gamma}{\partial m} k + (\gamma + i) \frac{\partial k}{\partial m} \quad (7.20)$$

Multiplying equation 7.19 by  $[r]$  and substituting equation 7.20 leads to the following complex scalar equation with the unknowns  $\frac{\partial \gamma}{\partial m}$  and  $\frac{\partial k}{\partial m}$ :

$$\left( r \frac{\partial \tilde{M}}{\partial m} q \right) p^2 + (r \tilde{M} q) 2p k \frac{\partial \gamma}{\partial m} + (r \tilde{M} q) 2p (\gamma + i) \frac{\partial k}{\partial m} + \left( r \frac{\partial \tilde{K}}{\partial m} q \right) - \left( r \frac{\partial \tilde{A}}{\partial k} q \right) \frac{\partial k}{\partial m} = 0 \quad (7.21)$$

As before,  $r \frac{\partial \tilde{M}}{\partial m} q$  is a short notation for the scalar quantity

$$[r] \left[ \frac{\partial \tilde{M}}{\partial m} \right] \{q\}, \text{ etc.}$$

No numerical evaluations have been made of methods to compute  $\frac{\partial \gamma}{\partial m}$ .

### 7.3 Derivatives of the Vibration Solution

It is customary, and for practical reasons often mandatory, to reduce the number of degrees of freedom of the flutter equation by modalization. In general the modalization is accomplished with the help of vibration modes.

During an optimization process, in which the structure is resized in several steps, updating of the vibration modes is often necessary. Every update requires the solution of the vibration equation.

Possibly the number of updates can be reduced if the change of vibration modes due to the changes in design variables is used. E.g., for the  $k^{\text{th}}$  mode:

$$\left\{ z_{k_{\text{new}}} \right\} = \left\{ z_{k_{\text{old}}} \right\} + \left[ \frac{\partial z_k}{\partial m_i} \right] \{\Delta m_i\} \quad (7.22)$$

The procedure of Section 7.2 is followed.

The vibration equation is:

$$\left[ - [M] \omega_k^2 + [K] \right] \{z\} = 0 \quad (7.23)$$

For a root  $\omega = \omega_k$ :

$$\left[ - [M] \omega_k^2 + [K] \right] \{z_k\} = 0 \quad (7.24)$$

and, because  $[M]$  and  $[K]$  are symmetric matrices:

$$\left[ z_k \right] \left[ - [M] \omega_k^2 + [K] \right] = 0 \quad (7.25)$$

Differentiation of equation 7.24 with respect to  $m_i$  gives:

$$\left[ - \left[ \frac{\partial M}{\partial m_i} \right] \omega_k^2 - [M] \frac{\partial \omega_k^2}{\partial m_i} + \left[ \frac{\partial K}{\partial m_i} \right] \right] \{z_k\} + \left[ - [M] \omega_k^2 + [K] \right] \left\{ \frac{\partial z_k}{\partial m_i} \right\} = 0 \quad (7.26)$$

Premultiplication by  $\left[ z_k \right]$  leads to a solution for  $\frac{\partial \omega_k^2}{\partial m_i}$ :

$$\frac{\partial \omega_k^2}{\partial m_i} = \frac{1}{\left[ z_k \right] [M] \{z_k\}} \left[ z_k \right] \left[ - \left[ \frac{\partial M}{\partial m_i} \right] \omega_k^2 + \left[ \frac{\partial K}{\partial m_i} \right] \right] \{z_k\} \quad (7.27)$$

Following the reasoning in Sections 7.1 and 7.2 the derivative of the vibration mode can be derived:

$$\left\{ \frac{\partial z_k}{\partial m_i} \right\} = - \left[ - [\underline{M}] \omega_k^2 + [\underline{K}] \right]^{-1} \left[ - \left[ \frac{\partial M}{\partial m_i} \right] \omega_k^2 - [\underline{M}] \frac{\partial \omega_k^2}{\partial m_i} + \left[ \frac{\partial K}{\partial m_i} \right] \right] \{ z_k \} \quad (7.28)$$

from which

$$\left\{ \frac{\partial z_k}{\partial m_i} \right\} = \left\{ \begin{array}{c} \frac{\partial z_k}{\partial m_i} \\ 0 \end{array} \right\} \quad (7.29)$$

A new modalization matrix is formed by assembling columns  $\{ z_{k_{\text{new}}} \}$ , for as many values of  $k$  as there are vibration modes to be used, into one matrix.

At any step in the optimization process the quantities defining  $\frac{\partial \omega_k^2}{\partial m_i}$  according to equation 7.27 are available. However, the inverse in equation 7.28 must be evaluated once for every mode and the total expression in equation 7.28 must be evaluated for every mode as well as every design variable. Whether, in view of this considerable number of computational operations this derivative method of updating the vibration modes is more efficient than repeating the characteristic value solution followed by a determination of the modal columns, seems doubtful and has not been investigated.

#### 7.4 Flutter Speed Derivatives with Variable Modes

When modalizing the flutter equation with natural vibration modes, these modes can be considered as a new set of coordinates suitable for defining approximate flutter solutions for configurations similar to the configuration for which the vibration modes are determined. In determining the flutter speed and flutter root derivatives these modes are then considered constant. The formulation of the derivatives in Sections 7.1 and 7.2 is based on the assumption of constant modalization matrices.

If, however, the use of updated vibration modes is considered an integral part of modalization, then the modalization matrix cannot be considered constant, and additional terms with derivatives of the modalization matrix must be included. This is pointed out in Reference 21. A procedure, based on the results of Section 7.3, is outlined in the following paragraphs.

Consider  $\left[ \frac{\partial M}{\partial m} \right]$  in equation 7.5. If equation 7.3 is a modalized flutter equation,  $[M]$  is a modalized mass matrix.

$$[M] = [\bar{z}^T] [\tilde{M}] [\bar{z}] \quad (7.30)$$

where  $[\tilde{M}]$  is the basic matrix and  $[\bar{z}]$  is a matrix of modal columns.

It follows:

$$\left[ \frac{\partial M}{\partial m} \right] = [\bar{z}^T] \left[ \frac{\partial \tilde{M}}{\partial m} \right] [\bar{z}] + \left[ \frac{\partial \bar{z}^T}{\partial m} \right] [\tilde{M}] [\bar{z}] + [\bar{z}^T] [\tilde{M}] \left[ \frac{\partial \bar{z}}{\partial m} \right] \quad (7.31)$$

The second and third term on the right hand side of equation 7.31 are extra terms due to letting the modes be functions of the design variables.

An equation similar to equation 7.31 can be written for  $\left[ \frac{\partial K}{\partial m} \right]$ .

The expression for  $\left[ \frac{\partial A}{\partial m} \right]$  becomes:

$$\left[ \frac{\partial A}{\partial m} \right] = [\bar{z}^T] \left[ \frac{\partial \tilde{A}}{\partial k} \right] [\bar{z}] \frac{\partial k}{\partial m} + [\bar{z}^T] [\tilde{A}] \left[ \frac{\partial \bar{z}}{\partial m} \right] + \left[ \frac{\partial \bar{z}^T}{\partial m} \right] [\tilde{A}] [\bar{z}] \quad (7.32)$$

The matrix  $\left[ \frac{\partial \bar{z}}{\partial m} \right]$  is formed by assembling columns  $\left\{ \frac{\partial z_k}{\partial m} \right\}$ , given by equation 7.29, for as many  $\{z_k\}$  as there are vibration modes used in the modalization.

Inclusion of  $\left[ \frac{\partial \bar{z}}{\partial m} \right]$  in the formulation of the flutter speed and flutter root derivatives is expected to lead to more accurate derivatives and consequently to fewer resizing steps in the optimization process. This, presumably, will reduce the number of vibration analyses required for remodalization, which is the same result as is expected from the procedure presented in Section 7.3.

That a potential for an increase in efficiency due to the inclusion of  $\left[ \frac{\partial \bar{z}}{\partial m} \right]$  exists, follows from the following consideration.

In choosing design variables, practical considerations may favor local stiffnesses as independent design variables. To reflect the effect of a local stiffness variation in the modalized flutter equation, the "resolution power" of the modes must be sufficient to recognize this stiffness variation. When the flutter speed and flutter root derivatives are determined under the

assumption of constant modes it seems self-evident that more modes are required for derivatives of a desired accuracy than when the modes are considered functions of the design variables. In the latter case, if the resolution of the unmodalized vibration equation is sufficient to recognize local stiffness variations, then this resolution power is partially

preserved through the inclusion of the  $\left[ \frac{\partial \bar{z}}{\partial m} \right]$  terms.

Numerical investigations to confirm the above expectations are recommended.

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The literature in this bibliography is arranged in the following groups:

1. Technical articles and reports dealing with structural optimization with flutter constraints. Although an attempt is made to make this list all inclusive, duplication of the same source material is avoided.
2. Technical articles and reports describing analytical techniques that may be useful in implementing efficient flutter optimization procedures.
3. Technical articles and reports dealing with structural optimization in general.
4. Books on optimization in general and books, Symposium Proceedings and other material on structural optimization. Any articles on flutter optimization in this material are explicitly referenced in Group 1.

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16. Abstract  The subject study addressed computational aspects of (1) flutter optimization (minimization of structural mass subject to specified flutter requirements), (2) methods for solving the flutter equations, and (3) efficient methods for computing generalized aerodynamic force coefficients in the repetitive analysis environment of computer-aided structural design. The principal results of the study are summarized in a companion report. The present report contains supplemental data and supporting information on various aspects of the study including the following:  (1) Details of a two-dimensional Regula Falsi approach to solving the generalized flutter equation are presented, (2) The method of Incremented Flutter Analysis and some of its applications are discussed, (3) The use of Velocity Potential Influence Coefficients in a five-matrix product formulation of the generalized aerodynamic force coefficients is delineated. Options for computational operations required to generate generalized aerodynamic force coefficients are compared in detail, (4) Theoretical considerations related to optimization with one or more flutter constraints are presented as well as practical experience with an actual structural design problem, (5) Expressions for derivatives of flutter-related quantities with respect to design variables are presented. Included are flutter-speed derivatives with variable modes.  A bibliography is included.					
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